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HOSOYA POLYNOMIAL, WIENER INDEX AND HYPER WIENER INDEX OF THE FAMILIES OF CORONA PRODUCT OF P_2 WITH P_n, W_n, S_n AND K_n

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Abstract

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The Hosoya polynomial of G is $H(G,x) = \sum_{\{u,v\}\subset V(G)} x^{d(u,v)}$, where $d(u,v)$ denotes the distance between the vertices u and v. In this research paper, we will determine the Hosoya polynomial of the families of Corona product of P_2 with P_n, S_n, W_n, and K_n. Moreover, we will determine the Wiener Index and hyper Wiener Index of the respective families.

1. Introduction

Let G be a connected graph, the vertex set and edge set of G is denoted by $V(G)$ and $E(G)$ respectively. The distance $d(u,v)$ between u and v is the length of the smallest path, where $u,v \in V(G)$. The maximum distance between the two vertices of a graph G is called the diameter of G and is denoted by $d(G)$. The degree of a vertex $u \in V(G)$ is the number of vertices joined to u, or the number of edges incident with u and is denoted by d_u. The Hosoya polynomial of a graph G is a generating function that indicates about the distribution of distance in a graph. This polynomial was introduced by a Japanese chemist Haruo Hosoya in 1988. Haruo Hosoya discovered a new formula for the Wiener Index in terms of graph distance and therefore this polynomial is known by the name of its discoverer. The Hosoya polynomial of a connected graph G is defined as (Hosoya, 1988):

$$H_G(x) = \sum_{k=1}^{l} d_k x^k$$

where l is the diameter of G and d_k is the number of paths of length k between the two vertices of G.

The most interesting application of Hosoya polynomials (Amin et al., 2017) is that almost all distance-based graph invariants, which are used to predict physical, chemical, and pharmacological properties of organic molecules can be recovered from Hosoya polynomials. In fact, it calculates the number of distances of paths of different lengths in the graph G (Amin et al., 2017).

In 1996, the Wiener polynomial was solitarily initiated and examined. In fact, the polynomial was originally known as the Wiener polynomial but later, under the admiration of the researcher the name was changed to Hosoya polynomial. The fringe benefit of the Hosoya polynomial is that it contains abundance of knowledge about graph invariants that are distance based. For example, the first derivative of the Hosoya polynomial at $x = 1$ is equal to the Wiener Index. This property of the Hosoya polynomial makes it phenomenal. The Hosoya polynomial gives a supplemental knowledge about distances in a graph G (Sagan et al., 1996).

The Hosoya polynomial of various chemical structures has been determined (Ali & Ali, 2011, Farahani, 2013 and Sadeghieh et al., 2017). Moreover, the Hosoya polynomial of some graph families have been examined (Farahani, 2015, Farahani, 2015, Narayankar et al., 2012). Also, the Hosoya polynomial of families of graphs has been studied (Stevanovic, 2001 and Wang et al., 2016). The graphs of which the Hosoya polynomial is to be determined can be seen in (Nazeer & Kousar, 2014) and (Nazeer & Kousar, 2014).

The Wiener Index (Rezai et al., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$W(G) = H'(G,x)|_{x=1}$$

The hyper Wiener Index (Rezai et al., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$WW(G) = H'(G,x)|_{x=1} + \frac{1}{2}H''(G,x)|_{x=1}$$

where the former and later are the first and second derivatives of the Hosoya polynomial at $x = 1$.

2. Materials and Methods

After going through a series of research papers that are based on the Hosoya polynomial of families of graphs, a simple calculation for finding out the Hosoya polynomial, Wiener Index and hyper Wiener Index will be put forward in order to understand these terminologies in a better way.

Create the first form of the understudy family of the graph, and then calculate the distance between the vertices of the graph. After calculating the distance, separate the vertices according to their degrees. Calculate $d(G,k)$ where k denotes the distance parameter. Now, after the completion of this process, revise the same steps for the second form of the family of the graph, then the third, and so on. Once several forms of graphs are checked, a formula is generated that can satisfy the number of paths calculated for each $d(G,k)$. Combining the terms will give the Hosoya polynomial of the family of the graph.

3. Results

In this section, we determine the Hosoya polynomial, Wiener Index and hyper Wiener Index of the families of Corona product of P_2 with P_n, S_n, W_n, and K_n.

Before proceeding, please consider the definition of the Corona product of graphs.

3.1 Definition

The Corona product (Nazeer & Kousar, 2014 and Nazeer & Kousar, 2014) of a graph G with a graph H, denoted by $G \odot H$, is a graph obtained by taking one copy of a n-vertex graph G and n copies $H_1, H_2, \dots, H_n$ of H and then joining the i^{th} vertex of G to every vertex in H_i.

Theorem 3.1: The Hosoya polynomial of the family of Corona product of P_2 with P_n for all integer number $n \geq 2$ is

$$H(P_2 \odot P_n, x) = (4n-1)x + (n^2 - n + 2)x^2 + n^2 x^3$$

The Wiener Index of the family is

$$W((P_2 \odot P_n) = 5n^2 + 2n + 3$$

The hyper Wiener Index is as follows

$$WW((P_2 \odot P_n) = 9n^2 + n + 5$$

Proof:

The diameter of this specific family is 3, so we will calculate all the edges paths from 1 till the diameter of the graph i.e. 3. First, we will partition the vertices of the graph $P_2 \odot P_n$. There are 4 vertices of degree 2, $2n-4$ vertices of degree 3 and 2 vertices of degree $n+1$. Thus, we have the following partition,

$$V_2 = \{v \in V(P_2 \odot P_n | d_v = 2\} \rightarrow |V_2| = 4 \quad (3.1.1)$$

$$V_3 = \{v \in V(P_2 \odot P_n | d_v = 3\} \rightarrow |V_3| = 2n-4 \quad (3.1.2)$$

$$V_{n+1} = \{v \in V(P_2 \odot P_n | d_v = n+1\} \rightarrow |V_{n+1}| = 2 \quad (3.1.3)$$

Thus, $V_2 \cup V_3 \cup V_{n+1}$ makes the vertex set of $P_2 \odot P_n$ and otherwise empty.

Figure 1: $P_2 \odot P_4$

Now we know that,

$$|E(G)| = \frac{1}{2}\sum_{k=\delta}^{\Delta} |V_k| \times k \quad (3.1.4)$$

where Δ and δ are the maximum and minimum of $d_v, v \in V(G)$, respectively, thus

$$|E(G)| = \frac{1}{2}[2 \times |V_2| + 3 \times |V_3| + (n+1) \times |V_{n+1}|] \quad (3.1.5)$$

Making substitutions from (3.1.1-3.1.3) in (3.1.5),

$$|E(G)| = \frac{1}{2}[2 \times 4 + 3 \times (2n-4) + (n+1) \times 2]$$
$$= \frac{1}{2}[8 + 6n - 12 + 2n + 2]$$

$$|E(G)| = 4n - 1 \quad (3.1.6)$$

Now to compute the Hosoya polynomial of the Corona product of $P_2 \odot P_n$, we will use the definition of the Hosoya polynomial from (Hosoya, 1988). Thus, we have

$$H(G, x) = \sum_{k=1}^{d(G)} d(G,k) x^k \quad (3.1.7)$$

where $d(G,k)$ is the representation of the distance $d(u,v)=k$ and $1 \leq k \leq diam(G)$. From the explanation and construction of the graph $P_2 \odot P_n$ we have the following calculations for $d(P_2 \odot P_n, k)$ where $1 \leq k \leq d(P_2 \odot P_n)$ for all k.

$$d(P_2 \odot P_n, 1) = |E(P_2 \odot P_n)| = 4n - 1 \quad (3.1.8)$$

$$d(P_2 \odot P_n, 2) = n^2 - n + 2 \quad (3.1.9)$$

The number of 2-edges paths between the vertices of V_2 and $V_{n+1} \subset V(P_2 \odot P_n)$ are 4. There are $2n-4$, 2-edges paths between the vertices of V_3 and V_{n+1}. The number of 2-edges paths between the vertices $u, v \in V_2$ are $n^2 - 3n + 2$. Adding all of these, we get the second sentence of the Hosoya polynomial which is of the form

$$[4 + 2n - 4 + n^2 - 3n + 2]x^2 = [n^2 - n + 2]x^2$$

$$d(P_2 \odot P_n, 3) = n^2 \quad (3.1.10)$$

There are no 3-edges paths between the vertices of V_2. The number of 3-edges paths between the vertices of V_2 and V_3 are n^2. The third and last sentence of the Hosoya polynomial is, $[n^2]x^3$.

Thus, the Hosoya polynomial of $P_2 \odot P_n$ $\forall n \geq 2$ is,

$$H(P_2 \odot P_n, x) = (4n-1)x + (n^2 - n + 2)x^2 + n^2x^3 \quad (3.1.11)$$

Taking the first derivative of equation (3.1.11) at $x = 1$ will provide the Wiener index. Thus, following in the below manner

$$W(P_2 \odot P_n) = (4n-1) + 2(n^2 - n + 2)x|_{x=1} + 3n^2x^2|_{x=1}$$

$$W(P_2 \odot P_n) = (4n-1) + 2(n^2 - n + 2) + 3n^2$$

$$W(P_2 \odot P_n) = 5n^2 + 2n + 3 \quad (3.1.12)$$

Now, to obtain the hyper Wiener index of this family, will take first the second derivative of the Hosoya polynomial at $x = 1$.

$$H''(P_2 \odot P_n) = 2(n^2 - n + 2) + 6n^2x|_{x=1} = 2(n^2 - n + 2) + 6n^2$$

$$H''(P_2 \odot P_n) = 8n^2 - 2n + 4 \quad (3.1.13)$$

Now, to obtain the final form of the hyper Wiener index, add the first derivative from (3.1.12) and half of the second derivative (3.1.13) of the Hosoya polynomial. Thus, we have

$$WW(P_2 \odot P_n) = 5n^2 + 2n + 3 + \frac{1}{2}(8n^2 - 2n + 4)$$

$$WW(P_2 \odot P_n) = 9n^2 + n + 5 \quad (3.1.14)$$

This completes the proof. For example, for $n = 4$, we have the following Hosoya polynomial, Wiener index and hyper Wiener index as follows:

$$H(P_2 \odot P_4) = 15x + 14x^2 + 16x^3$$

$$W(P_2 \odot P_4) = 91$$

$$WW(P_2 \odot P_4) = 153$$

Theorem 3.2: The Hosoya polynomial of the family of Corona product of P_2 with S_n for all integer number $n \geq 2$ is

$$H(P_2 \odot S_n, x) = (4n+3)x + (n^2 + n + 2)x^2 + (n^2 + 2n + 1)x^3$$

The Wiener index of this family is

$$W((P_2 \odot S_n) = 5n^2 + 12n + 10$$

The hyper Wiener Index is as follows

$$WW((P_2 \odot S_n) = 9n^2 + 21n + 15$$

Proof:

The diameter of this specific family is 3, so we will calculate all the edges paths from 1 till the diameter of the graph i.e. 3. First, we will partition the vertices of the graph $P_2 \odot S_n$. There are $2n$ vertices of degree 2, 2 vertices of degree $n+1$ and 2 vertices of degree $n+2$.

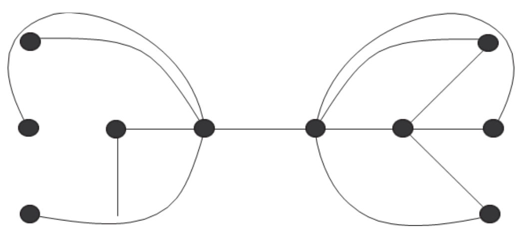

Figure 2: $P_2 \odot S_3$

According to steps as in Theorem 3.1, the number of edges are $4n+3$. Now, we will determine the Hosoya polynomial of this family of graph. From the explanation and construction of the graph $P_2 \odot S_n$ we have the following calculations for $d(P_2 \odot S_n, k)$ where $1 \le k \le d(P_2 \odot S_n)$ for all k.

$$d(P_2 \odot S_n, 1) = |E(P_2 \odot S_n)| = 4n + 3 \quad (3.2.1)$$

$$d(P_2 \odot S_n, 2) = n^2 + n + 2 \quad (3.2.2)$$

The number of 2-edges paths between the vertices of $u, v \in V_2$ are $n^2 - n$. There are $2n$, 2-edges paths from V_2 to V_{n+2} and the number of 2-edges paths starting from V_{n+1} to V_{n+2} are 2. Thus, the second sentence of the Hosoya polynomial then becomes,

$$[n^2 - n + 2n + 2]x^2 = [n^2 + n + 2]x^2$$

$$d(P_2 \odot S_n, 3) = n^2 + 2n + 1 \quad (3.2.3)$$

There are n^2, 3-edges paths between the vertices of V_2. The number of 3-edges paths that starts from V_2 to V_{n+1} are $2n$ and there is 1, 3-edges path between the vertices $u, v \in V_{n+1}$. Thus, the third and last sentence of the Hosoya polynomial is, $[n^2 + 2n + 1]x^3$.

Thus, the Hosoya polynomial of the family of Corona product $P_2 \odot S_n \; \forall n \ge 2$ is,

$$H(P_2 \odot S_n, x) = (4n + 3)x + (n^2 + n + 2)x^2 + (n^2 + 2n + 1)x^3 \quad (3.2.4)$$

Following the same procedure as in Theorem 3.1, listed below is the Wiener index and hyper Wiener index of this respective family of graphs.

$$W(P_2 \odot S_n) = 5n^2 + 12n + 10 \quad (3.2.5)$$

$$WW(P_2 \odot S_n) = 9n^2 + 21n + 15 \quad (3.2.6)$$

This completes the proof. For example, for $n = 3$, we have the following Hosoya polynomial, Wiener index and hyper Wiener index as follows:

$$H(P_2 \odot S_3) = 15x + 14x^2 + 16x^3$$

$$W(P_2 \odot S_3) = 91$$

$$WW(P_2 \odot S_3) = 159$$

Theorem 3.3: The Hosoya polynomial of the family of Corona product of P_2 with W_n for all integer number $n \ge 3$ is

$$H(P_2 \odot W_n, x) = (6n + 3)x + (n^2 - n + 2)x^2 + (n^2 + 2n + 1)x^3$$

The Wiener index of this family is

$$W((P_2 \odot W_n) = 5n^2 + 10n + 10$$

The hyper Wiener Index is as follows

$$WW((P_2 \odot W_n) = 9n^2 + 15n + 15$$

Proof

The diameter of this specific family is 3, so we will calculate all the edges paths from 1 till the diameter of the graph i.e. 3. First, we will partition the vertices of the graph $P_2 \odot W_n$. There are $2n$ vertices of degree 4, 2 vertices of degree $n+1$ and 2 vertices of degree $n+2$.

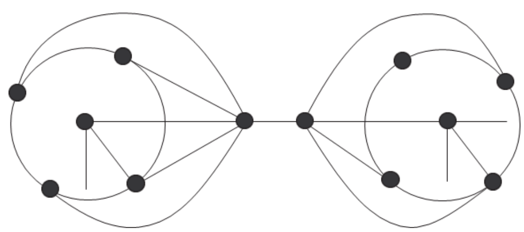

Figure 3: $P_2 \odot W_4$

The numbers of edges as performed in Theorem 3.1 are $6n+3$ and now we will determine the Hosoya polynomial of this family of graph. From the explanation and construction of the graph $P_2 \odot W_n$ we have the following calculations for $d(P_2 \odot W_n, k)$ where $1 \leq k \leq d(P_2 \odot W_n)$ for all k.

$$d(P_2 \odot W_n, 1) = |E(P_2 \odot W_n)| = 6n+3 \quad (3.3.1)$$

$$d(P_2 \odot W_n, 2) = n^2 - n + 2 \quad (3.3.2)$$

The number of 2-edges paths between the vertices of $u, v \in V_4$ are $n^2 - 3n$. There are 2, 2-edges paths starting from V_{n+1} to V_{n+2} and the number of 2-edges paths starting from V_4 to V_{n+2} are 2. Thus, the second sentence of the Hosoya polynomial then becomes,

$$[n^2 - 3n + 2 + 2n]x^2 = [n^2 - n + 2]x^2$$

$$d(P_2 \odot W_n, 3) = n^2 + 2n + 1 \quad (3.3.3)$$

There are n^2, 3-edges paths between the vertices of V_4. The number of 3-edges paths that starts from V_4 to V_{n+1} are n^2 and there is 1, 3-edges path between the vertices $u, v \in V_{n+1}$. Thus, the third and last sentence of the Hosoya polynomial is, $[n^2 + 2n + 1]x^3$.

Thus, the Hosoya polynomial of the family of Corona product $P_2 \odot W_n$ $\forall n \geq 3$ is,

$$H(P_2 \odot W_n, x) = (6n+3)x + (n^2 - n + 2)x^2 + (n^2 + 2n + 1)x^3 \quad (3.3.4)$$

Following the similar steps to compute the Wiener and hyper Wiener index of this family as executed in Theorem 3.1. Thus, we have

$$W(P_2 \odot W_n) = 5n^2 + 10n + 10 \quad (3.3.5)$$

$$WW(P_2 \odot W_n) = 9n^2 + 15n + 15 \quad (3.3.6)$$

This completes the proof. For example, for $n = 4$, we have the following Hosoya polynomial, Wiener index and hyper Wiener index as follows:

$$H(P_2 \odot W_4) = 27x + 14x^2 + 25x^3$$

$$W(P_2 \odot W_4) = 130$$

$$WW(P_2 \odot W_4) = 219$$

Theorem 3.4: The Hosoya polynomial of the family of Corona product of P_2 with K_n for all integer number $n \geq 3$ is

$$H(P_2 \odot K_n, x) = (n^2 + n + 1)x + 2nx^2 + n^2x^3$$

The Wiener index of this family is

$$W((P_2 \odot K_n) = 4n^2 + 5n + 1$$

The hyper Wiener Index is as follows

$$WW((P_2 \odot K_n) = 7n^2 + 7n + 1$$

Proof

The diameter of this specific family is 3, so we will calculate all the edges paths from 1 till the diameter of the graph i.e. 3. First, we will partition the vertices of the graph $P_2 \odot K_n$. There are $2n$ vertices of degree n and 2 vertices of degree $n+1$.

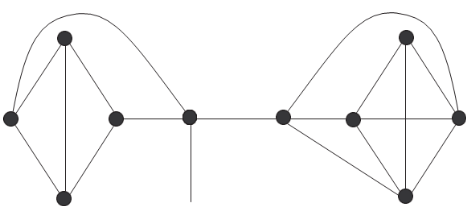

Figure 4: $P_2 \odot K_4$

The numbers of edges as performed in Theorem 3.1 are n^2+n+1 and now we will determine the Hosoya polynomial of this family of graph. From the explanation and construction of the graph $P_2 \odot K_n$ we have the following calculations for $d(P_2 \odot K_n, k)$ where $1 \le k \le d(P_2 \odot K_n)$ for all k.

$$d(P_2 \odot K_n, 1) = |E(P_2 \odot K_n)| = n^2+n+1 \quad (3.4.1)$$

$$d(P_2 \odot K_n, 2) = 2n \quad (3.4.2)$$

The number of 2-edges paths between the vertices of V_n to V_{n+1} are $2n$. Thus, the second sentence of the Hosoya polynomial then becomes, $[2n]x^2$

$$d(P_2 \odot K_n, 3) = n^2 \quad (3.4.3)$$

There are n^2, 3-edges paths between the vertices of V_n. Thus, the third and last sentence of the Hosoya polynomial is, $[n^2]x^3$.

Thus, the Hosoya polynomial of the family of Corona product $P_2 \odot K_n \ \forall n \ge 3$ is,

$$H(P_2 \odot K_n, x) = (n^2+n+1)x + 2nx^2 + n^2x^3 \quad (3.4.4)$$

To obtain the Wiener and hyper Wiener index of this family, we will proceed as shown in Theorem 3.1. Thus, the desired results are as follows

$$W(P_2 \odot K_n) = 4n^2+5n+1 \quad (3.4.5)$$

$$WW(P_2 \odot K_n) = 7n^2+7n+1 \quad (3.4.6)$$

This completes the proof. For example, for $n=4$, we have the following Hosoya polynomial, Wiener index and hyper Wiener index as follows:

$$H(P_2 \odot K_4) = 21x + 8x^2 + 16x^3$$

$$W(P_2 \odot K_4) = 85$$

$$WW(P_2 \odot K_4) = 141$$

4. Discussion

Graph theory is a valuable field of mathematics that has broad series of applications in many areas of science such as biology, chemistry, computer science, medicines, and electrical engineering etc. Graph theory has made it possible to study difficult structures and networks by transforming it into molecular graphs. In this paper, the researcher have determined the Hosoya polynomial, Wiener Index and hyper Wiener Index of the Corona product of families of P_2 with P_n, S_n, W_n and K_n.

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