Contents lists available http://www.kinnaird.edu.pk/

Journal of Natural and Applied Sciences Pakistan

Journal homepage: http://jnasp.kinnaird.edu.pk/

HOSOYA POLYNOMIAL, WIENER INDEX AND HYPER WIENER INDEX OF SOME FAMILIES OF GRAPHS OF DIAMETER 3

Fatima Aamir[1*], Dr. Imrana Kousar[1], Dr. Saima Nazeer [1]
[1]Department of Mathematics, Lahore College for Women University, Jail Road, Lahore, Pakistan

Article Info

*Corresponding Author
Email: fatimaaamirch_2@hotmail.com

Keywords
Distance, Hosoya Polynomial, Wiener Index, Hyper Wiener Index

Abstract

Let G be a simple connected graph having vertex set $V(G)$ and edge set $E(G)$. The Hosoya polynomial of G is $H(G,x) = \sum_{\{u,v\}\subset V(G)} x^{d(u,v)}$, where $d(u,v)$ denotes the distance between the vertices u and v. In this research paper, we will determine the Hosoya polynomial of the families of graphs of diameter 3. Moreover, we will determine the Wiener Index and hyper Wiener Index of the respective families.

1. Introduction

Let G be a connected graph, the vertex set and edge set of G is denoted by $V(G)$ and $E(G)$ respectively. The distance $d(u,v)$ between u and v is the length of the smallest path, where $u, v \in V(G)$. The maximum distance between the two vertices of a graph G is called the diameter of G and is denoted by $d(G)$. The degree of a vertex $u \in V(G)$ is the number of vertices joined to u or the number of edges incident with u and is denoted by d_u. The Hosoya polynomial of a graph G is a generating function that indicates about the distribution of distance in a graph. The polynomial was introduced by a Japanese chemist Haruo Hosoya in 1988. Haruo Hosoya discovered a new formula for the Wiener Index in terms of graph distance and therefore this polynomial is known by the name of its discoverer. The Hosoya polynomial of a connected graph G is defined as (Hosoya, 1988):

$$H(G,x) = \frac{1}{2} \sum_{v \in V(G)} V(G) \sum_{u \in V(G)} V(G)\ d(u,v)$$

The Hosoya polynomial of various chemical structures has been determined (Ali & Ali, 2011, Farahani, 2013 and Sadeghieh *et al*., 2017). Moreover, the Hosoya polynomial of some graph families have been examined (Farahani, 2015, Farahani, 2015, Narayankar *et al*., 2012). Also, the Hosoya polynomial of families of graphs has been studied (Stevanovic, 2001, Stevanovic & Gutman, 1999 and Wang et al., 2016). The Wiener Index (Rezai *et al*., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$W(G) = \frac{\partial H(G,x)}{\partial x}|_{x=1}$$

The hyper Wiener Index (Rezai et al., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$WW(G) = H'(G,x)|_{x=1} + \frac{1}{2}H''(G,x)|_{x=1}$$

where the former and later are the first and second derivatives of the Hosoya polynomial at $x = 1$.

1.1 Definition:

The graph (Sweetly & Joseph, 2007) Bistar $B(n,m)$ is obtained by joining the centers of $K_{1,n}$ and $K_{1,m}$.

1.2 Definition:

The graph (Sweetly & Joseph, 2007) $G(K_n^c)$ is obtained by joining n vertices to a single vertex of G.

1.3 Definition:

The graph (Sweetly & Joseph, 2007) $G \circ K_n^c$ is obtained by joining n vertices to each vertex of G.

1.4 Definition:

The graph (Sweetly & Joseph, 2007) $G(K_3)$ is obtained by joining n triangles to a single vertex of G.

1.5 Definition:

A planar graph (Sweetly & Joseph, 2007) whose edge set E can be decomposed into two disjoint subsets T and C, $E = T \cup C$, $T \cap C = \emptyset$, where the subgraph of $G(2,n)$ initiated by T is a tree with one vertex u of degree 2, one vertex v of degree n, u and v being adjacent and the rest of the $2 + n - 2 = n$ vertices of degree one and the subgraph generated by C is a cycle of length $2 + n - 2$ passing through all vertices of $G(2,n)$ except u and v, is an exceptional Halin graph. The graph consists of $n + 2$ vertices and $2n + 1$ edges, $\forall n \geq 6$.

1.6 Definition:

The Book graph (Murugan, 2015) is the Cartesian product of the complete bipartite graph $K_{1,n}$ ($\forall n \geq 1$) and the complete graph K_2.

1.7 Definition:

The Diamond graph (Ahmad & Ghemeci, 2017) is obtained by taking the Cartesian product of the path graphs P_2 and P_n ($\forall n \geq 3$), then adding one vertex at the top i.e. v and one vertex at the bottom i.e. w and joining each vertex $\{x_i, y_i: 1 \leq i \leq n\}$ with v and w.

2. Materials and Methods

A simple calculation for finding out the Hosoya polynomial, Wiener Index and hyper Wiener Index will be put forward in order to understand these.

3. Results

In this section, we determine the Hosoya polynomial, Wiener Index and hyper Wiener Index of some families of graphs of Diameter 3.

Theorem 3.1: The Hosoya polynomial of the Bistar graph $B(n,m)$, for all integer numbers $n,m \geq 2$ is

$$H(B(n,m),x) = (n+m+1)x + \frac{n^2+m^2+n+m}{2}x^2 + nmx^3$$

The Wiener Index of the family of Bistar graph is

$$W\big(B(n,m)\big) = n^2+m^2+2n+2m+3nm + 1$$

The hyper Wiener Index of the respective family is

$$WW\big(B(n,m)\big) = \frac{3n^2}{2}+\frac{3m^2}{2}+\frac{5n}{2}+\frac{5m}{2} + 6nm + 1$$

Proof:

Let $G = B(n,m)$ be a graph $\forall n,m \geq 2$ with $n+m+2$ vertices. First, we will partition the vertices according to their degrees respectively. There are $n+m$ vertices of degree 1, 1 vertex of degree $n+1$ and 1 vertex of degree $m+1$. The vertex set $V(B(n,m))$ is as follows

$$V_1 = \{v \in V(G) | d_v = 1\} \rightarrow |V_1| = n+m \quad (3.1.1)$$

$$V_{n+1} = \{v \in V(G) | d_v = n+1\} \rightarrow |V_{n+1}| = 1 \quad (3.1.2)$$

$$V_{m+1} = \{v \in V(G) | d_v = m+1\} \rightarrow |V_{m+1}| = 1 \quad (3.1.3)$$

Figure 1: $B(2,4)$

Now we know that,

$$|E(G)| = \frac{1}{2}\sum_{k=\delta}^{\Delta}|V_k| \times k \quad (3.1.4)$$

where Δ and δ are the maximum and minimum of $d_v, v \in V(G)$, respectively, thus

$$|E(G)| = \frac{1}{2}[1\times|V_1| + (n+1) \times |V_{n+1}| + (m+1) \times |V_{m+1}|] \quad (3.1.5)$$

Making substitutions from (3.1.1-3.1.3) in (3.1.5),

$$|E(G)| = \frac{1}{2}[1\times(n+m) + (n+1)\times 1 + (m+1)\times 1]$$

$$|E(G)| = n+m+1 \quad (3.1.6)$$

Now to compute the Hosoya polynomial of $B(n,m)$, we will use the definition of the

Hosoya polynomial from (Hosoya, 1988). Thus, we have

$$H(G,x)=\sum_{k=1}^{d(G)} d(G,k)x^k \tag{3.1.7}$$

where $d(G,k)$ is the representation of the distance $d(u,v)=k$ and $1\le k\le diam(G)$. From the explanation and construction of the graph $B(n,m)$ we have the following calculations for $d(B(n,m),k)$ where $1\le k\le d(B(n,m))$ for all k.

$$d(B(n,m),1)=|E(B(n,m))| = n+m+1 \tag{3.1.8}$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[n+m+1]x$.

$$d(B(n,m),2)=\frac{n^2+m^2+n+m}{2} \tag{3.1.9}$$

The total number of 2-edges paths between the vertices $u,v\in V_1$ are $\frac{n^2+m^2-n-m}{2}$. There are n, 2-edges paths from V_1 to V_{m+1} and m, 2-edges paths from V_1 to V_{n+1}. Hence, the second sentence of the Hosoya polynomial is $\left[\frac{n^2+m^2-n-m}{2}+n+m\right]x^2=[\frac{n^2+m^2+n+m}{2}]x^2$.

$$d(B(n,m),3)=nm \tag{3.1.10}$$

The total number of 3-edges paths between the vertices $u,v\in V_1$are nm. Thus, the third and last sentence of the Hosoya polynomial is $[nm]x^3$.

From equations $(3.1.8-3.1.10)$, we obtain the Hosoya polynomial of this family, i.e.

$$\begin{aligned}&H(B(n,m),x)\\&=(n+m+1)x\\&+\frac{n^2+m^2+n+m}{2}x^2+nmx^3\end{aligned} \tag{3.1.11}$$

Taking the first derivative of equation (3.1.11) at $x=1$ will provide the Wiener index. Thus, following in the below manner

$$\begin{aligned}W(B(n,m))=(n+m+1)&\\+(n^2+m^2+n+m)x|_{x=1}&\\+3nmx^2|_{x=1}&\end{aligned}$$

$$\begin{aligned}W(B(n,m))=(n+m+1)&\\+(n^2+m^2+n+m)+3nm&\end{aligned}$$

$$\begin{aligned}W(B(n,m))=n^2+m^2+2n+2m&\\+3nm+1&\end{aligned} \tag{3.1.12}$$

Now, to obtain the hyper Wiener index of this family, will take first the second derivative of the Hosoya polynomial at $x=1$.

$$\begin{aligned}H^{"}(B(n,m))&=(n^2+m^2+n+m)\\&+6nmx|_{x=1}\\&=(n^2+m^2+n+m)+6nm\end{aligned}$$

$$\begin{aligned}H^{"}(B(n,m))=n^2+m^2+6nm+n&\\+m&\end{aligned} \tag{3.1.13}$$

Now, to obtain the final form of the hyper Wiener index, add the first derivative from (3.1.12) and half of the second derivative (3.1.13) of the Hosoya polynomial. Thus, we have

$$\begin{aligned}WW(B(n,m))&=n^2+m^2+2n+2m+3nm\\&+1+\frac{1}{2}(n^2+m^2+6nm+n\\&+m)\end{aligned}$$

$$WW(B(n,m)) = \frac{3n^2}{2} + \frac{3m^2}{2} + \frac{5n}{2} + \frac{5m}{2} + 6nm + 1 \quad (3.1.14)$$

This completes the proof. For example, for $n = 2, m = 4$, we have the following Hosoya polynomial, Wiener index and hyper Wiener index as follows:

$$H(B(2,4)) = 7x + 13x^2 + 8x^3$$

$$W(B(2,4)) = 57$$

$$WW(2,4) = 94$$

Theorem 3.2: The Hosoya polynomial of $C_4(K_n^c), \forall n \geq 1$ is

$$H(C_4(K_n^c), x) = (n+4)x + \frac{n^2+3n+4}{2}x^2 + nx^3$$

The Wiener Index of this family is

$$W\big(C_4(K_n^c)\big) = n^2 + 7n + 8$$

The hyper Wiener Index is

$$WW\big(C_4(K_n^c)\big) = \frac{3n^2}{2} + \frac{23n}{2} + 10$$

Proof:

By using the definition of $G(K_n^c)$, one can see that there are n vertices of degree 1, 3 vertices of degree 2 and 1 vertex of degree $n+2$. Therefore, the total number of vertices of this family are $n+4$.

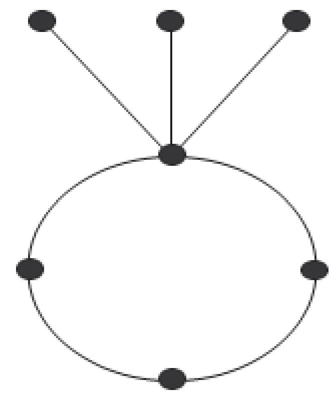

Figure 2: $C_4(K_3^c)$

Proceeding in the same manner as illustrated in Theorem 3.1, the number of edges obtained are $n+4$ and now we will determine the Hosoya polynomial of this graph.

$$d(G,1) = |E(G)| = n+4 \quad (3.2.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[n+4]x$.

$$d(G,2) = \frac{n^2+3n+4}{2} \quad (3.2.2)$$

There is 1, 2-edges path from V_{n+2} to V_1 and 1, 2-edges path between the vertices $u, v \in V_2$. The total number of 2-edges paths between V_1 and V_2 are $2n$. There are $\frac{n^2-n}{2}$, 2-edges paths between the vertices $u, v \in V_1$. Hence, the second sentence of the Hosoya polynomial is $[\frac{n^2+3n+4}{2}]x^2$.

$$d(G,3) = n \quad (3.2.3)$$

The total number of 3-edges paths between the vertices $V_1$and $V_2$are n. Thus, the third and last sentence of the Hosoya polynomial is $[n]x^3$.

Thus, the Hosoya polynomial of this family, i.e.

$$H(C_4(K_n^c),x) = (n+4)x + \frac{n^2+3n+4}{2}x^2 + nx^3 \qquad (3.2.4)$$

According to the steps performed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W\big(C_4(K_n^c)\big) = n^2 + 7n + 8 \qquad (3.2.5)$$

$$WW\big(C_4(K_n^c)\big) = \frac{3n^2}{2} + \frac{23n}{2} + 10 \qquad (3.2.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_4(K_4^c),x) = 8x + 16x^2 + 4x^3$$

$$W\big(C_4(K_4^c)\big) = 52$$

$$WW\big(C_4(K_4^c)\big) = 80$$

Theorem 3.3: The Hosoya polynomial of $C_5(K_n^c), \forall n \geq 1$ is

$$H(C_5(K_n^c),x) = (n+5)x + \frac{n^2+3n+10}{2}x^2 + 2nx^3$$

The Wiener Index of this family is

$$W\big(C_5(K_n^c)\big) = n^2 + 10n + 15$$

The hyper Wiener Index is

$$WW\big(C_5(K_n^c)\big) = \frac{3n^2}{2} + \frac{35n}{2} + 20$$

Proof:

By means of the definition of $G(K_n^c)$, one can see that there are n vertices of degree 1, 4 vertices of degree 2 and 1 vertex of degree $n+2$. Hence, there are $n+5$ vertices in this whole family of graphs

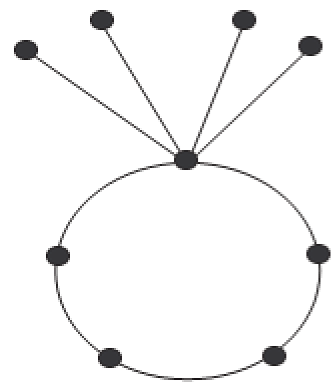

Figure 3: $C_5(K_4^c)$

Continuing in the same way as executed in Theorem 3.1, the number of edges obtained are $n+5$ and now we will establish the Hosoya polynomial of this graph.

$$d(G,1) = |E(G)| = n+5 \qquad (3.3.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[n+5]x$.

$$d(G,2) = \frac{n^2+3n+10}{2} \qquad (3.3.2)$$

There are 2, 2-edges path from V_{n+2} to V_2 and 3, 2-edges path between the vertices $u,v \in V_2$. The total number of 2-edges paths between V_1 and V_2 are $2n$. There are $\frac{n^2-n}{2}$, 2-edges paths between the vertices $u,v \in V_1$. Hence, the second sentence of the Hosoya polynomial is $[\frac{n^2+3n+10}{2}]x^2$.

$$d(G,3) = 2n \tag{3.3.3}$$

The total number of 3-edges paths between the vertices V_1 and V_2 are $2n$. Thus, the third and last sentence of the Hosoya polynomial is $[2n]x^3$.

Thus, the Hosoya polynomial of this family, i.e.

$$H(C_5(K_n^c), x) = (n+5)x + \frac{n^2+3n+10}{2}x^2 + 2nx^3 \tag{3.3.4}$$

According to the steps performed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W\big(C_5(K_n^c)\big) = n^2 + 10n + 15 \tag{3.3.5}$$

$$WW\big(C_5(K_n^c)\big) = \frac{3n^2}{2} + \frac{35n}{2} + 20 \tag{3.3.6}$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_5(K_4^c), x) = 10x + 25x^2 + 10x^3$$

$$W\big(C_5(K_4^c)\big) = 90$$

$$WW\big(C_5(K_4^c)\big) = 145$$

Theorem 3.4: The Hosoya polynomial of $K_m \circ K_n^c$, $\forall m, n \geq 3$ is

$$H(K_m \circ K_n^c, x) = \frac{2nm + m^2 - m}{2}x + \frac{m(2mn - 3n + n^2)}{2}x^2 + \frac{m^2 - m}{2}n^2x^3$$

The Wiener Index is

$$W(K_m \circ K_n^c) = \frac{3n^2m^2}{2} + 2m^2n - \frac{n^2m}{2} + \frac{m^2}{2} - \frac{m}{2} - 2nm$$

The hyper Wiener Index is

$$WW(K_m \circ K_n^c) = 3n^2m^2 + 3m^2n - \frac{3n^2m}{2} + \frac{m^2}{2} - \frac{m}{2} - \frac{7nm}{2}$$

Proof:

By making use of the definition of $G \circ K_n^c$, one can see that there are nm vertices of degree 1 and m vertices of degree $n + m - 1$. Hence, the total number of vertices of this family are $m(n+1)$.

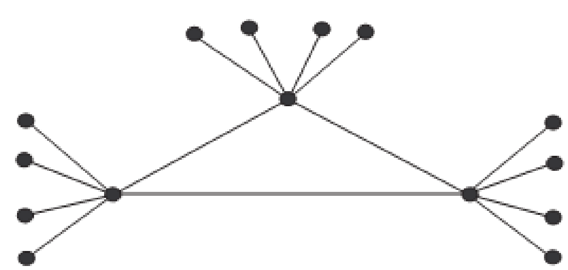

Figure 4: $K_3 \circ K_4^c$

The number of edges are $\frac{2nm+m^2-m}{2}$, according to the steps performed in Theorem 3.1. Now, we will examine the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = \frac{2nm + m^2 - m}{2} \quad (3.4.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{2nm+m^2-m}{2}\right]x$.

$$d(G,2) = \frac{m(2mn - 3n + n^2)}{2} \quad (3.4.2)$$

There are $nm(m-1)$, 2-edges path from V_{n+m-1} to V_1 and $\frac{m}{2}(n^2-n)$, 2-edges path between the vertices $u, v \in V_2$. Hence, the second sentence of the Hosoya polynomial is $\left[nm(m-1) + \frac{m(n^2-n)}{2}\right]x^2 =$ $\left[\frac{m(2mn-3n+n^2)}{2}\right]x^2$.

$$d(G,3) = \frac{m^2 - m}{2}n^2 \quad (3.4.3)$$

The total number of 3-edges paths between the vertices $u, v \in V_1$are $\frac{m^2-m}{2}n^2$. Thus, the third and last sentence of the Hosoya polynomial is $\left[\frac{m^2-m}{2}n^2\right]x^3$.

Thus, the Hosoya polynomial of this family, i.e.

$$\begin{aligned} H(K_m \circ K_n^c, x) &= \frac{2nm + m^2 - m}{2}x \\ &+ \frac{m(2mn - 3n + n^2)}{2}x^2 \\ &+ \frac{m^2 - m}{2}n^2x^3 \end{aligned} \quad (3.4.4)$$

According to the steps performed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$\begin{aligned} W(K_m \circ K_n^c) &= \frac{3n^2m^2}{2} + 2m^2n \\ &- \frac{n^2m}{2} + \frac{m^2}{2} - \frac{m}{2} \\ &- 2nm \end{aligned} \quad (3.4.5)$$

$$\begin{aligned} WW(K_m \circ K_n^c) \\ &= 3n^2m^2 + 3m^2n \\ &- \frac{3n^2m}{2} + \frac{m^2}{2} - \frac{m}{2} \\ &- \frac{7nm}{2} \end{aligned} \quad (3.4.6)$$

This completes the proof. For example, for $n = 3, m = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(K_3 \circ K_4^c, x) = 15x + 42x^2 + 48x^3$$

$$W(K_3 \circ K_4^c) = 243$$

$$WW(K_3 \circ K_4^c) = 429$$

Theorem 3.5: The Hosoya polynomial of $G = C_4(K_3)$ obtained by taking n copies of K_3 attached at a single vertex of C_4, $\forall n \geq 1$ is

$$\begin{aligned} H(C_4(K_3), x) &= (3n + 4)x + 2(n^2 + n + 1)x^2 \\ &+ 2nx^3 \end{aligned}$$

The Wiener Index is

$$W\big(C_4(K_3)\big) = 4n^2 + 13n + 8$$

The hyper Wiener Index is

$$WW\big(C_4(K_3)\big) = 6n^2 + 21n + 10$$

Proof:

By means of the definition of this specific family of graphs, we can see that there are $2n+3$ vertices of degree 2 and 1 vertex of degree $2n+2$. Hence, the total number of vertices that accompany this family are $2n+4$.

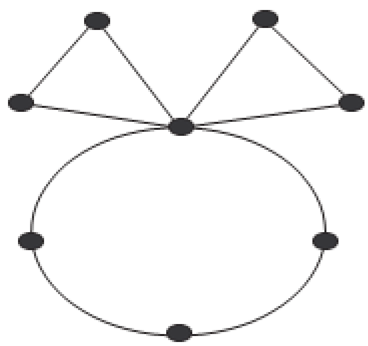

Figure 5.
$C_4(K_3), n = 2$

The number of edges are $3n+4$. Now, we will determine the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = 3n + 4 \qquad (3.5.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[3n+4]x$.

$$d(G,2) = 2(n^2 + n + 1) \qquad (3.5.2)$$

There is 1, 2-edges path from V_2 to V_{2n+2} and n^2+2n+1, 2-edges path between the vertices $u, v \in V_2$. Hence, the second sentence of the Hosoya polynomial is $2[n^2+n+1]x^2$.

$$d(G,3) = 2n \qquad (3.5.3)$$

The total number of 3-edges paths between the vertices $u, v \in V_2$are $2n$. Thus, the third and last sentence of the Hosoya polynomial is $[2n]x^3$.

Thus, the Hosoya polynomial of this family, i.e.

$$H(C_4(K_3), x) = (3n+4)x + 2(n^2+n+1)x^2 + 2nx^3 \qquad (3.5.4)$$

According to the steps performed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W\big(C_4(K_3)\big) = 4n^2 + 13n + 8 \qquad (3.5.5)$$

$$WW\big(C_4(K_3)\big) = 6n^2 + 21n + 10 \qquad (3.5.6)$$

This completes the proof. For example, for $n = 2$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_4(K_3), x) = 10x + 14x^2 + 8x^3$$

$$W\big(C_4(K_3)\big) = 50$$

$$WW\big(C_4(K_3)\big) = 76$$

Theorem 3.6: The Hosoya polynomial of $G = C_4(K_n), \forall n \geq 3$ is

$$\square(C_4(K_n), x) = \frac{n^2 - n + 8}{2}x + 2nx^2 + (n - 1)x^3$$

The Wiener Index is

$$W\big(C_4(K_n)\big) = \frac{n^2 + 13n + 2}{2}$$

The hyper Wiener Index is

$$WW\big(C_4(K_n)\big) = \frac{n^2 + 23n - 4}{2}$$

Proof:

The graph $G = C_4(K_n)$ is obtained by attaching a single copy of the complete graph K_n with one vertex of C_4. We can see that there are 3 vertices of degree 2, $n-1$ vertices of degree $n-1$ and 1 vertex of degree $n+1$. Hence, we get a total of $n+3$ vertices.

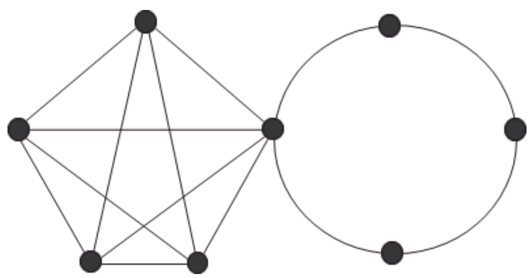

Figure 6: $C_4(K_5)$

The number of edges are $\frac{n^2-n+8}{2}$. Now, we will examine the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = \frac{n^2 - n + 8}{2} \qquad (3.6.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{n^2-n+8}{2}\right]x$.

$$d(G,2) = 2n \qquad (3.6.2)$$

There is 1, 2-edges path between the vertices $u, v \in V_2$ and 1, 2-edges path from V_{n+1} to V_2. The total number of 2-edges paths from V_2 to V_{n-1} are $2n-2$. Hence, the second sentence of the Hosoya polynomial is $[2n]x^2$.

$$d(G,3) = n - 1 \qquad (3.6.3)$$

The total number of 3-edges paths from the vertices V_2 to V_{n-1} are $n-1$. Thus, the third and last sentence of the Hosoya polynomial is $[n-1]x^3$.

Thus, the Hosoya polynomial of this family is,

$$H(C_4(K_n), x) = \frac{n^2 - n + 8}{2}x + 2nx^2 + (n-1)x^3 \qquad (3.6.4)$$

The following are the Wiener and hyper Wiener Index of this family determined after following the steps illustrated in Theorem 3.1.

$$W\big(C_4(K_n)\big) = \frac{n^2 + 13n + 2}{2} \qquad (3.6.5)$$

$$WW\big(C_4(K_n)\big) = \frac{n^2 + 23n - 4}{2} \qquad (3.6.6)$$

This completes the proof. For example, for $n = 5$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_4(K_5), x) = 14x + 20x^2 + 12x^3$$

$$W\big(C_4(K_5)\big) = 46$$

$$WW\big(C_4(K_5)\big) = 68$$

Theorem 3.7: The Hosoya polynomial of $G = C_5(K_n), \forall n \geq 3$ is

$$H(C_5(K_n), x) = \frac{n^2 - n + 10}{2}x + (2n+3)x^2 + (2n-2)x^3$$

The Wiener Index is

$$W\big(C_5(K_n)\big) = \frac{n^2 + 19n + 10}{2}$$

The hyper Wiener Index is

$$WW\big(C_5(K_n)\big) = \frac{n^2 + 35n + 4}{2}$$

Proof:

The graph $G = C_5(K_n)$ is obtained by attaching a single copy of the complete graph K_n with one vertex of C_5. There are 4 vertices of degree 2, $n-1$ vertices of degree $n-1$ and 1 vertex of degree $n+1$.

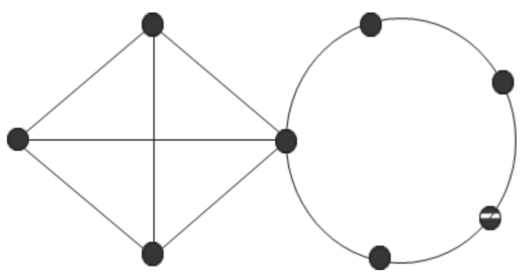

Figure 7: $C_5(K_4)$

There are $n+4$ vertices and edges $\frac{n^2-n+10}{2}$. Now, we will examine the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = \frac{n^2 - n + 10}{2} \quad (3.7.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{n^2-n+10}{2}\right]x$.

$$d(G,2) = 2n + 3 \quad (3.7.2)$$

There are 3, 2-edges path between the vertices $u, v \in V_2$ and 2, 2-edges path between the vertices of V_{n+1} and V_2. The total number of 2-edges paths from V_2 to V_{n-1} are $2n-2$. Hence, the second sentence of the Hosoya polynomial is $[2n+3]x^2$.

$$d(G,3) = 2n - 2 \quad (3.7.3)$$

The total number of 3-edges paths from the vertices V_2 to V_{n-1} are $2n-2$. Thus, the third and last sentence of the Hosoya polynomial is $[2n-2]x^3$.

Thus, the Hosoya polynomial of this family is,

$$H(C_5(K_n),x) = \frac{n^2 - n + 10}{2}x + (2n + 3)x^2 + (2n-2)x^3 \quad (3.7.4)$$

According to the steps performed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W\big(C_5(K_n)\big) = \frac{n^2 + 19n + 10}{2} \quad (3.7.5)$$

$$WW\big(C_5(K_n)\big) = \frac{n^2 + 35n + 4}{2} \quad (3.7.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_5(K_4),x) = 11x + 11x^2 + 6x^3$$

$$W\big(C_5(K_4)\big) = 51$$

$$WW\big(C_5(K_4)\big) = 80$$

Theorem 3.8: The Hosoya polynomial of $G = C_4(K_3) + e, \forall n \geq 1$ where e represents an extra edge between the two vertices of C_4 and is obtained by taking n copies of K_3 attached to a single vertex of C_4 is

$$H(C_4(K_3) + e, x) = (3n + 5)x + (2n^2 + 2n + 1)x^2 + 2nx^3$$

The Wiener Index is

$$W(C_4(K_3) + e) = 4n^2 + 13n + 7$$

The hyper Wiener Index is

$$WW(C_4(K_3) + e) = 6n^2 + 21n + 8$$

Proof:

The construction of this family is similar to $C_4(K_3)$ and we have only added an extra edge between the two vertices of C_4 to obtain this family of graphs. We can see that there are $2n + 1$ vertices of degree 2, 2 vertices of degree 3 and 1 vertex of degree $n + 2$. Hence, we get a total of $2n + 4$ vertices.

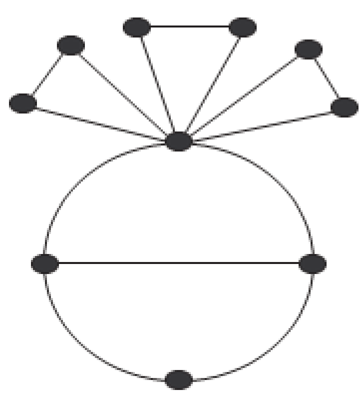

Figure 8: $C_4(K_3) + e, n = 3$

Now, we will examine the Hosoya polynomial of this family.

$$d(G, 1) = |E(G)| = 3n + 5 \qquad (3.8.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[3n + 5]x$.

$$d(G, 2) = 2n^2 + 2n + 1 \qquad (3.8.2)$$

There are $2n^2 - 2n$, 2-edges path between the vertices $u, v \in V_2$. The total number of 2-edges paths from V_{2n+2} to V_2 are 1. Also, there are $4n$, 2-edges paths from V_3 to V_2. Hence, the second sentence of the Hosoya polynomial is $[2n^2 + 2n + 1]x^2$.

$$d(G, 3) = 2n \qquad (3.8.3)$$

The total number of 3-edges paths from the vertices $u, v \in V_2$ to are $2n$. Thus, the third and last sentence of the Hosoya polynomial is $[2n]x^3$.

Hence, the Hosoya polynomial is

$$H(C_4(K_3) + e, x) = (3n + 5)x + (2n^2 + 2n + 1)x^2 + 2nx^3 \qquad (3.8.4)$$

The following are the Wiener and hyper Wiener Index of this family,

$$W(C_4(K_3) + e) = 4n^2 + 13n + 7 \qquad (3.8.5)$$

$$WW(C_4(K_3) + e) = 6n^2 + 21n + 8 \qquad (3.8.6)$$

This completes the proof. For example, for $n = 3$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_4(K_3) + e, x) = 14x + 25x^2 + 6x^3$$

$$W(C_4(K_3) + e) = 82$$

$$WW(C_4(K_3) + e) = 125$$

Theorem 3.9: The Hosoya polynomial of $G = \square_4(K_n) + e, \forall n \geq 3$ where e represents an extra edge between the two vertices of C_4 and is obtained by attaching a single copy of K_n attached to a single vertex of C_4 is

$$H(C_4(K_n) + e, x) = \frac{n^2 - n + 10}{2}x + (2n-1)x^2 + (n-1)x^3$$

The Wiener Index is

$$W(C_4(K_n) + e) = \frac{\square^2 + 13n}{2}$$

The hyper Wiener Index is

$$WW(C_4(K_n) + e) = \frac{n^2 + 23n - 8}{2}$$

Proof:

The construction of this family is similar to $C_4(K_n)$ and we have only added an extra edge between the two vertices of C_4 to obtain this family of graphs. We can see that there is 1 vertex of degree 2, 2 vertices of degree 3, $n-1$ vertices of degree $n-1$ and 1 vertex of degree $n+1$. Hence, we get a total of $n+3$ vertices.

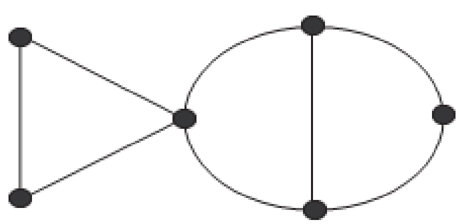

Figure 9: $C_4(K_3)$

Now, we will examine the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = \frac{n^2 - n + 10}{2} \quad (3.9.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{n^2-n+10}{2}\right]x$.

$$d(G,2) = 2n - 1 \quad (3.9.2)$$

The total number of 2-edges paths from V_3 to V_{n-1} are $2n-2$. Also, there is 1, 2-edges paths from V_{n+1} to V_2. Hence, the second sentence of the Hosoya polynomial is $[2n-1]x^2$.

$$d(G,3) = n - 1 \quad (3.9.3)$$

The number of 3-edges paths between the vertices V_2 and V_{n-1} to are $n-1$. Thus, the third and last sentence of the Hosoya polynomial is $[n-1]x^3$.

Thus, the calculated Hosoya polynomial of this family is

$$H(C_4(K_n) + e, x) = \frac{n^2 - n + 10}{2}x + (2n-1)x^2 + (n-1)x^3 \quad (3.9.4)$$

Below are the determined Wiener and hyper Wiener Index of this family,

$$W(C_4(K_n) + e) = \frac{n^2 + 13n}{2} \quad (3.9.5)$$

$$WW(C_4(K_n) + e) = \frac{n^2 + 23n - 8}{2} \quad (3.9.6)$$

This completes the proof. For example, for $n = 3$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(C_4(K_3) + e, x) = 8x + 5x^2 + 2x^3$$

$$W(C_4(K_3) + e) = 24$$

$$WW(C_4(K_3) + e) = 35$$

Theorem 3.10: The Hosoya polynomial of $G = K_n$ joined to one copy of K_n by an edge, $\forall n \geq 3$ is

$$H(G,x) = (n^2 - n + 1)x + (2n - 2)x^2 + (n^2 - 2n + 1)x^3$$

The Wiener Index of this respective family is

$$W(G) = 4n^2 - 3n$$

The hyper Wiener Index is

$$WW(G) = 7n^2 - 7n + 1$$

Proof:

As from the explanation of G from the statement of the theorem, we can see that are 2 vertices of degree n and $2n - 2$ vertices of degree $n - 1$. Thus, there are $2n$ vertices.

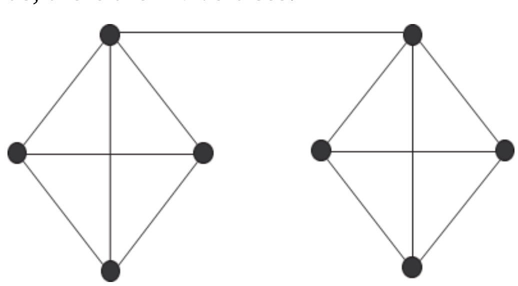

Figure 10: K_4 joined to one copy of K_4

Now, we will examine the Hosoya polynomial of this family.

$$d(G, 1) = |E(G)| = n^2 - n + 1 \qquad (3.10.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[n^2 - n + 1]x$.

$$d(G, 2) = 2n - 2 \qquad (3.10.2)$$

There are $2n - 2$, 2-edges paths from V_n to V_{n-1}. Hence, the second sentence of the Hosoya polynomial is $[2n - 2]x^2$.

$$d(G, 3) = n^2 - 2n + 1 \qquad (3.10.3)$$

The total number of 3-edges paths from the vertices $u, v \in V_{n-1}$ to are $n^2 - 2n + 1$. Thus, the third and last sentence of the Hosoya polynomial is $[n^2 - 2n + 1]x^3$.

The computed Hosoya polynomial is,

$$H(G,x) = (n^2 - n + 1)x + (2n - 2)x^2 + (n^2 - 2n + 1)x^3 \qquad (3.10.4)$$

Listed below are the Wiener and hyper Wiener Index of this family,

$$W(G) = 4n^2 - 3n \qquad (3.10.5)$$

$$WW(G) = 7n^2 - 7n + 1 \qquad (3.10.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G,x) = 13x + 6x^2 + 9x^3$$

$$W(G) = 52$$

$$WW(G) = 85$$

Theorem 3.11: The Hosoya polynomial of $G = K_n$ joined to two copies of K_n by an edge, $\forall n \geq 3$ is

$$H(G,x) = \frac{3n^2 - 3n}{2}x + (2n^2 - 4n + 2)x^2 + (n^2 - 2n + 1)x^3$$

The Wiener Index of this respective family is

$$W(G) = \frac{17n^2 - 31n + 14}{2}$$

The hyper Wiener Index is

$$WW(G) = \frac{27n^2 - 51n + 24}{2}$$

Proof:

As from the details of G in the statement of the theorem, we can see that are $3n - 4$ vertices of degree $n - 1$ and 2 vertices of degree $2n - 2$. Thus, there are $3n - 2$ vertices.

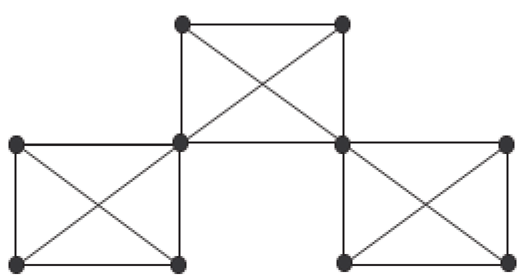

Figure 11: K_4 joined to two copies of K_4

Now, we will work out the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = \frac{3n^2 - 3n}{2} \quad (3.11.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{3n^2-3n}{2}\right]x$.

$$d(G,2) = 2n^2 - 4n + 2 \quad (3.11.2)$$

There are $2n^2 - 6n + 4$, 2-edges paths between the vertices $u, v \in V_{n-1}$. There are $2n - 2$, 2-edges paths from V_{2n-2} to V_{n-1}. Hence, the second sentence of the Hosoya polynomial is $[2n^2 - 4n + 2]x^2$.

$$d(G,3) = n^2 - 2n + 1 \quad (3.11.3)$$

The total number of 3-edges paths from the vertices $u, v \in V_{n-1}$ to are $n^2 - 2n + 1$. Thus, the third and last sentence of the Hosoya polynomial is $[n^2 - 2n + 1]x^3$.

Thus, the Hosoya polynomial is

$$H(G,x) = \frac{3n^2 - 3n}{2}x + (2n^2 - 4n + 2)x^2 + (n^2 - 2n + 1)x^3 \quad (3.11.4)$$

The following are the Wiener and hyper Wiener Index of this family,

$$W(G) = \frac{17n^2 - 31n + 14}{2} \quad (3.11.5)$$

$$WW(G) = \frac{27n^2 - 51n + 24}{2} \quad (3.11.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G,x) = 18x + 18x^2 + 9x^3$$

$$W(G) = 81$$

$$WW(G) = 126$$

Theorem 3.12: The Hosoya polynomial of $G = K_n$ joined with P_3 at a single vertex, $\forall n \geq 3$ is

$$H(G, x) = \frac{n^2 - n + 4}{2} x + nx^2 + (n-1)x^3$$

The Wiener Index of this respective family is

$$W(G) = \frac{n^2 + 9n - 2}{2}$$

The hyper Wiener Index is

$$WW(G) = \frac{n^2 + 17n - 8}{2}$$

Proof:

We have 1 vertex of degree 1, 1 vertex of degree 2, $n-1$ vertices of degree $n-1$ and 1 vertex of degree n. Thus, there are $n+2$ vertices.

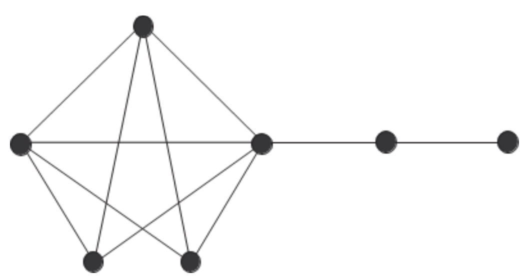

Figure 12: K_5 joined P_3 at a single vertex

Now, we will examine the Hosoya polynomial of this family.

$$d(G, 1) = |E(G)| = \frac{n^2 - n + 4}{2} \qquad (3.12.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{n^2-n+4}{2}\right] x$.

$$d(G, 2) = n \qquad (3.12.2)$$

Between the vertices of V_{n-1} and V_1, the number of 2-edges paths are $n-1$. There is 1, 2-edges paths from V_n to V_1. Hence, the second sentence of the Hosoya polynomial is $[n]x^2$.

$$d(G, 3) = n - 1 \qquad (3.12.3)$$

The total number of 3-edges paths between V_{n-1} and V_1 to are $n-1$. Thus, the third and last sentence of the Hosoya polynomial is $[n-1]x^3$.

The determined Hosoya polynomial is,

$$H(G, x) = \frac{n^2 - n + 4}{2} x + nx^2 + (n - 1)x^3 \qquad (3.12.4)$$

According to the steps executed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W(G) = \frac{n^2 + 9n - 2}{2}$$

$$WW(G) = \frac{n^2 + 17n - 8}{2}$$

This completes the proof. For example, for $n = 5$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G, x) = 12x + 5x^2 + 4x^3$$

$$W(G) = 34$$

$$WW(G) = 51$$

Theorem 3.13: The Hosoya polynomial of the graph G obtained by attaching a pendant vertex of $K_{1,m}, \forall m \geq 2$ with a vertex of $K_n, \forall n \geq 3$ is

$$H(G,x) = \frac{n^2 - n + 2m}{2}x + \frac{m^2 - m + 2n - 2}{2}x^2 + (m-1)(n-1)x^3$$

The Wiener Index is

$$W(G) = \frac{n^2}{2} + m^2 - \frac{3n}{2} - 3m + 3mn + 1$$

The hyper Wiener Index is

$$WW(G) = \frac{n^2 + 3m^2 - 7n - 13m + 12nm + 6}{2}$$

Proof:

This family of graph have $m-1$ vertices of degree 1, 1 vertex of degree m, $n-1$ vertices of degree $n-1$ and 1 vertex of degree n. Thus, there are total $m+n$ vertices.

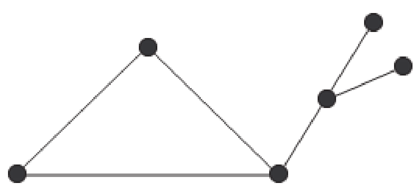

Figure 13. A pendant vertex of $K_{1,3}$ attached with a vertex of K_3

Now, we will calculate the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = \frac{n^2 - n + 2m}{2} \qquad (3.13.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{n^2-n+2m}{2}\right]x$.

$$d(G,2) = \frac{m^2 - m + 2n - 2}{2} \qquad (3.13.2)$$

There are $n-1$, 2-edges paths from V_{n-1} to V_m. The number of 2-edges paths between $u, v \in V_1$ are $\frac{m^2-3m+2}{2}$ and between the vertices of V_n and V_1, the number of 2-edges paths are $m-1$. Hence, the second sentence of the Hosoya polynomial is $[\frac{m^2-m+2n-2}{2}]x^2$.

$$d(G,3) = (m-1)(n-1) \qquad (3.13.3)$$

The total number of 3-edges paths between V_1 and V_{n-1} to are $(m-1)(n-1)$. Thus, the third and last sentence of the Hosoya polynomial is $[(m-1)(n-1)]x^3$.

Consequently, the Hosoya polynomial of this family is,

$$H(G,x) = \frac{n^2 - n + 2m}{2}x + \frac{m^2 - m + 2n - 2}{2}x^2 + (m-1)(n-1)x^3 \qquad (3.13.4)$$

So, the following are the Wiener and hyper Wiener Index of this family,

$$W(G) = \frac{n^2}{2} + m^2 - \frac{3n}{2} - 3m + 3mn + 1 \qquad (3.13.5)$$

$$WW(G) = \frac{n^2 + 3m^2 - 7n - 13m + 12nm + 6}{2} \qquad (3.13.6)$$

This completes the proof. For example, for $m = 3, n = 3$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G, x) = 6x + 5x^2 + 4x^3$$

$$W(G) = 28$$

$$WW(G) = 45$$

Theorem 3.14: The Hosoya polynomial of the Halin graph $G(2, n), \forall n \geq 6$ is

$$H(G(2, n), x) = (2n + 1)x + \frac{n^2 - 3n + 10}{2}x^2 + (n - 5)x^3$$

The Wiener Index is

$$W\big(G(2, n)\big) = n^2 + 2n - 4$$

The hyper Wiener Index is

$$WW\big(G(2, n)\big) = \frac{3n^2 + 7n - 28}{2}$$

Proof:

From the structure of the Halin graph, one can make a note that there is 1 vertex of degree 2, n vertices of degree 3 and 1 vertex of degree n. Hence, there are total $n + 2$ vertices.

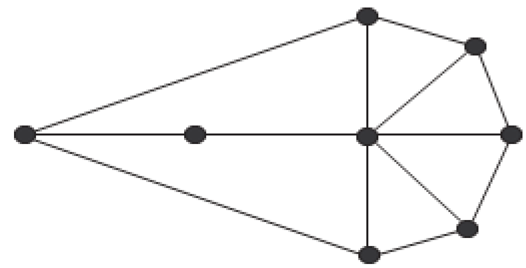

Figure 14: Halin graph $G(2, 6)$

We will determine the Hosoya polynomial of this family.

$$d(G(2, n), 1) = |E(G(2, n))| = 2n + 1 \qquad (3.14.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[2n + 1]x$.

$$d(G(2, n), 2) = \frac{n^2 - 3n + 10}{2} \qquad (3.14.2)$$

Between the vertices V_3 to V_n, the number of 2-edges paths is 1. From V_3 to V_2 there are $n - 1$ and between the vertices $u, v \in V_3$, the number of 2-edges paths are $\frac{n^2-5n+10}{2}$. Hence, the second sentence of the Hosoya polynomial is $[\frac{n^2-3n+10}{2}]x^2$.

$$d(G(2, n), 3) = n - 5 \qquad (3.14.3)$$

The total number of 3-edges paths between $u, v \in V_3$ to are $n - 5$. Thus, the third and last sentence of the Hosoya polynomial is $[n - 5]x^3$.

The Hosoya polynomial of this family calculated is,

$$H(G(2,n),x) = (2n+1)x + \frac{n^2-3n+10}{2}x^2 + (n-5)x^3 \quad (3.14.4)$$

Now, the following are the Wiener and hyper Wiener Index of this family,

$$W\big(G(2,n)\big) = n^2 + 2n - 4 \quad (3.14.5)$$

$$WW\big(G(2,n)\big) = \frac{3n^2+7n-28}{2} \quad (3.14.6)$$

This completes the proof. For example, for $n = 6$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G(2,6),x) = 13x + 14x^2 + x^3$$

$$W(G(2,6)) = 44$$

$$WW(G(2,6)) = 61$$

Theorem 3.15: The Hosoya polynomial of the graph obtained by joining n pendant vertices to one of the vertex of C_3 and 1 pendant edge is subdivided is

$$H(G,x) = (n+4)x + \frac{n^2+3n+2}{2}x^2 + (n+1)x^3$$

The Wiener Index is

$$W(G) = n^2 + 7n + 7$$

The hyper Wiener Index is

$$WW(G) = \frac{3n^2+23n+22}{2}$$

Proof:

It is clear from the figure of the graph that the number of vertices of degree 1 are n, 3 vertices of degree 2 and 1 vertex of degree $n+2$. Hence, there are $n+4$ vertices.

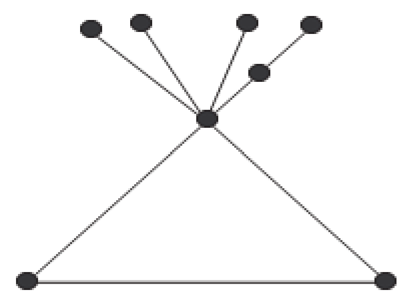

Figure 15: 4 pendant vertices are joined to one vertex of C_3 and one pendant edge is subdivided

Now, we will examine the Hosoya polynomial of this family.

$$d(G,1) = |E(G)| = n + 4 \quad (3.15.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[n+4]x$.

$$d(G,2) = \frac{n^2+3n+2}{2} \quad (3.15.2)$$

There is 1, 2-edges path between V_1 and V_{n+2}. The number of 2-edges paths between $u, v \in V_2$ are 2 and between the vertices V_1 and V_2, are $3n-3$. Between the vertices $u, v \in V_1$ there are $\frac{n^2-3n+2}{2}$. Hence, the second sentence of the Hosoya polynomial is $[\frac{n^2+3n+2}{2}]x^2$.

$$d(G,3) = n + 1 \quad (3.15.3)$$

The total number of 3-edges paths between $u, v \in V_1$ to are $n+1$. Thus, the third and last sentence of the Hosoya polynomial is $[n+1]x^3$.

Thus, the Hosoya polynomial of this family is,

$$H(G,x) = (n+4)x + \frac{n^2+3n+2}{2}x^2 + (n+1)x^3 \quad (3.15.4)$$

According to the steps executed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W(G) = n^2 + 7n + 7 \quad (3.15.5)$$

$$WW(G) = \frac{3n^2+23n+22}{2} \quad (3.15.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G,x) = 8x + 15x^2 + 15x^3$$

$$W(G) = 51$$

$$WW(G) = 81$$

Theorem 3.16: The Hosoya polynomial of the Book graph $K_{1,n} \times K_2, \forall n \geq 1$ is

$$H(K_{1,n} \times K_2, x) = (3n+1)x + (n^2+n)x^2 + (n^2-n)x^3$$

The Wiener Index is

$$W(K_{1,n} \times K_2) = 5n^2 + 2n + 1$$

The hyper Wiener Index is

$$WW(K_{1,n} \times K_2) = 9n^2 + 1$$

Proof:

It is clear from the figure of the Book graph, that there are $2n$ vertices of degree 2 and 2 vertices of degree $n+1$. Thus, there are $2n+2$ vertices in the graph. Observance shows that there are $3n+1$ number of edges. Now, we will examine the Hosoya polynomial for this particular family of graph.

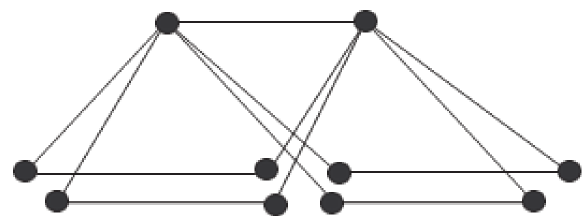

Figure 16: Book graph $K_{1,4} \times K_2$

Now, we will calculate the Hosoya polynomial of the Book graph. Here $G = K_{1,n} \times K_2$.

$$d(G,1) = |E(G)| = 3n + 1 \quad (3.16.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[3n+1]x$.

$$d(G,2) = n^2 + n \quad (3.16.2)$$

There are $2n$, 2-edges path between V_{n+1} and V_2. The number of 2-edges paths between $u, v \in V_2$ are $n^2 - n$. Hence, the second sentence of the Hosoya polynomial is $[n^2+n]x^2$.

$$d(G,3) = n^2 - n \quad (3.16.3)$$

The total number of 3-edges paths between $u, v \in V_2$ to are $n^2 - n$. Thus, the third and last sentence of the Hosoya polynomial is $[n^2 - n]x^3$.

The determined Hosoya polynomial is,

$$H(K_{1,n} \times K_2, x) = (3n+1)x + (n^2+n)x^2 + (n^2 - n)x^3 \quad (3.16.4)$$

The Wiener and hyper Wiener Index of this family are as following,

$$W(K_{1,n} \times K_2) = 5n^2 + 2n + 1 \quad (3.16.5)$$

$$WW(K_{1,n} \times K_2) = 9n^2 + 1 \quad (3.16.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(K_{1,4} \times K_2, x) = 13x + 20x^2 + 12x^3$$

$$W(K_{1,4} \times K_2) = 89$$

$$WW(K_{1,4} \times K_2) = 145$$

Theorem 3.17: The Hosoya polynomial of the Diamond graph $\forall n \geq 3$ is

$$H(G, x) = (5n-2)x + 2nx^2 + (4n-9)x^3$$

The Wiener Index is

$$W(G) = 21n - 29$$

The hyper Wiener Index is

$$WW(G) = 35n - 56$$

Proof:

One can see from the figure and definition from (Ahmad & Ghemeci, 2017) that there are 4 vertices of degree 3, $2n - 4$ vertices of degree 4 and 2 vertices of degree n. This constitutes to the the total number of vertices which are $2n + 2$ in number. The total number of edges are $5n - 2$. In order to get the Hosoya polynomial of the graph, we will proceed as follows:

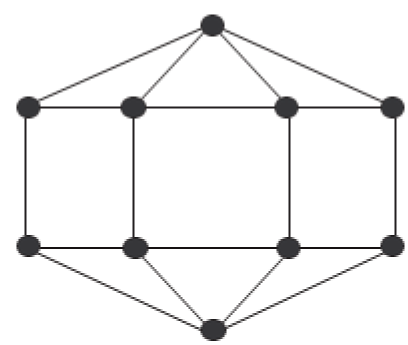

Figure 17: Diamond graph, $n = 4$

$$d(G, 1) = |E(G)| = 5n - 2 \quad (3.17.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $[5n - 2]x$.

$$d(G, 2) = 2n \quad (3.17.2)$$

The number of 2-edges paths between the vertices $u, v \in V_3$ and $u, v \in V_4$ are 2 and $2n - 6$. From the vertices of V_3 to V_n, from V_4 to V_n and from V_3 to V_4, the number of 2-edges paths are $4, 2n - 4$ and 4. Hence, the second sentence of the Hosoya polynomial is $[2n]x^2$.

$$d(G,3) = 4n - 9 \quad (3.17.3)$$

From the vertices of V_4 to V_3 or vice versa, the number of 3-edges paths $4n - 12$. Between the vertices $u, v \in V_n$ and $u, v \in V_3$, the number of 3-edges paths are 1 and 2. Thus, the third and last sentence of the Hosoya polynomial is $[4n - 9]x^3$.

Thus, the Hosoya polynomial of this family is,

$$H(G,x) = (5n-2)x + 2nx^2 + (4n - 9)x^3 \quad (3.17.4)$$

Moreover, the Wiener and hyper Wiener Index of this family are,

$$W(G) = 21n - 29 \quad (3.17.5)$$

$$WW(G) = 35n - 56 \quad (3.16.6)$$

This completes the proof. For example, for $n = 4$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G,x) = 18x + 8x^2 + 7x^3$$

$$W(G) = 55$$

$$WW(G) = 84$$

Theorem 3.18: The Hosoya polynomial of the graph obtained by attaching the complete graph $K_n, \forall n \geq 3$ with a single vertex of the Wheel graph $W_m, \forall m \geq 4$ is

$$H(G,x) = \frac{n^2 - n + 4m}{2}x + \frac{m^2 - 3m + 6n - 6}{2}x^2 + (n-1)(m-3)x^3$$

The Wiener Index is

$$W(G) = \frac{n^2}{2} + m^2 - \frac{7n}{2} - 4m + 3nm + 3$$

The hyper Wiener Index is

$$WW(G) = \frac{n^2 + 3m^2 - 19n - 17m + 12nm + 18}{2}$$

Proof:

From the figure, one can note that there are $m - 1$ vertices of degree 3, $n - 1$ vertices of degree $n - 1$, 1 vertex of degree m and 1 vertex of degree $n + 2$. Thus, according to this partitioning there are total $n + m$ vertices in this family of graphs. By observing, we come to a conclusion that there are $\frac{n^2-n+4m}{2}$ edges. Now, we will compute the Hosoya polynomial.

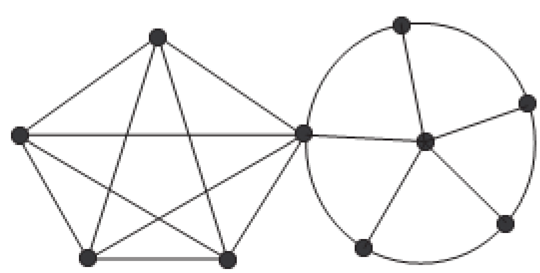

Figure 18: K_5 attached with a single vertex of the Wheel graph W_5

$$d(G,1) = |E(G)| = \frac{n^2 - n + 4m}{2} \quad (3.18.1)$$

The total number of 1-edges paths are equal to the total number of edges. Thus, the first sentence of the Hosoya polynomial is $\left[\frac{n^2-n+4m}{2}\right]x$.

$$d(G,2) = \frac{m^2 - 3m + 6n - 6}{2} \tag{3.18.2}$$

There are $2n-2$, 2-edges paths from V_{n-1} to V_3, $n-1$ from V_{n-1} to V_m and $m-3$ from V_3 to V_{n+2}. The number of 2-edges paths between the vertices of V_3 are $\frac{m^2-5m+6}{2}$Hence, the second sentence of the Hosoya polynomial is $[\frac{m^2-3m+6n-6}{2}]x^2$.

$$d(G,3) = (n-1)(m-3) \tag{3.18.3}$$

From the vertices of V_{n-1} to V_3 or vice versa, the number of 3-edges paths $(n-1)(m-3)$. Thus, the third and last sentence of the Hosoya polynomial is $[(n-1)(m-3)]x^3$.

Hence, the analyzed Hosoya polynomial of this family is,

$$\begin{aligned} &H(G,x) \\ &= \frac{n^2 - n + 4m}{2}x \\ &+ \frac{m^2 - 3m + 6n - 6}{2}x^2 \\ &+ (n-1)(m-3)x^3 \end{aligned} \tag{3.18.4}$$

According to the steps performed in Theorem 3.1, the following are the Wiener and hyper Wiener Index of this family,

$$W(G) = \frac{n^2}{2} + m^2 - \frac{7n}{2} - 4m + 3nm + 3 \tag{3.18.5}$$

$$WW(G) = \frac{n^2 + 3m^2 - 19n - 17m + 12nm + }{2} \tag{3.18.6}$$

This completes the proof. For example, for $n = 5, m = 5$, we have the following calculated Hosoya polynomial, Wiener and hyper Wiener index:

$$H(G,x) = 20x + 17x^2 + 8x^3$$

$$W(G) = 78$$

$$WW(G) = 119$$

4. Discussion

In this paper, we have determined the Hosoya polynomial, Wiener and hyper Wiener Index of some families of Graphs of Diameter 3.

5. References

Ahmad, A. & Ghemeci, R. M. (2017). Radio Labeling of some Ladder-Related Graphs. Math.Reports. 19(69), 1, 107-119.

Ali, A. & Ali, A. M. (2011). Hosoya Polynomial of Pentachains. Match Commun. Math. Comput. Chem. 65, 807-819.

Farahani, M. R. (2015). The Wiener Index and Hosoya Polynomial of a class of Jahangir graphs $J_{3,m}$. Fundamental Journal of Mathematics and Mathematical Sciences. 3(1), 91-96.

Farahani, M. R. (2015). Hosoya Polynomial of Jahangir Graphs $J_{4,m}$. Global Journal of Mathematics. 3(1), 232-236.

Farahani, M. R. (2013). On the Schultz polynomial and Hosoya polynomial of Circumcoronene Series of Benzenoid. J. Appl. Math. and Informatics. 31(5-6), 595-608.

Hosoya, H. (1988). On some counting polynomials in chemistry. Discrete Applied Mathematics. 19, 239-257.

Murugan, M. (2015). Radio Number for Special Family of Graphs of Diameter 2, 3 and 4. International J. Math. Combin. 3, 99-104.

Narayankar, K. P., B., L. S., Mathad, V. & Gutman, I. (2012). Hosoya polynomial of Hanoi Graphs. Kragujevac. Journal of Mathematics. 36(1), 51-57.

Rezai, M., Farahani, M. R., Khalid, W. & Baig, A. Q. (2017). Computation of Hosoya Polynomial, Wiener and Hyper Wiener Index of Jahangir Graph $J_{6,m}$. Journal of Prime Research in Mathematics. 13, 30-40.

Sadeghieh, A., Alikhani, S., Ghanbari, N. & Khalaf, A. J. M. (2017). Hosoya polynomial of some cactus chains. Cogent Mathematics. 4.

Stevanovic, D. (2011). Hosoya Polynomial of composite graphs. Discrete Mathematics. 235, 237-244.

Stevanovic, D. & Gutman, I. (1999). Hosoya polynomial of trees with upto 11 vertices. Collection of Scientific Papers of the Faculty of Science Kragujevac. 21, 111-119.

Sweetly, R. & Joseph, J. P. (2007). Radio Number of Graphs of Diameter Three. Journal of Discrete Mathematical Sciences and Cryptography. 10(5), 721-730.

Wang, S., Farahani, M. R., Kanna, M. R. R., Jamil, M. K. & Kumar, R. P. (2016). The Wiener Index and the Hosoya Polynomial of the Jahangir Graphs. Applied and Computational Mathematics. 5(3), 138-141.