Contents lists available http://www.kinnaird.edu.pk/

Journal of Natural and Applied Sciences Pakistan

Journal homepage: http://jnasp.kinnaird.edu.pk/

HOSOYA POLYNOMIAL OF CARTESIAN PRODUCT OF CYCLES $C_m \times C_n$, $(\forall m \geq n)$ m BEING EVEN & n BEING ODD

Fatima Aamir[1*], Dr. Imrana Kousar[1], Dr. Saima Nazeer [1]
[1]Department of Mathematics, Lahore College for Women University, Jail Road, Lahore, Pakistan

Article Info

*Corresponding Author
Email: fatimaaamirch_2@hotmail.com

Keywords
Distance, Hosoya Polynomial, Wiener Index, Hyper Wiener Index, Cartesian Product

Abstract

Let G be a simple connected graph having vertex set $V(G)$ and edge set $E(G)$. The Hosoya polynomial of G is $H(G,x) = \sum_{\{u,v\}\subset V(G)} x^{d(u,v)}$, where $d(u,v)$ denotes the distance between the vertices u and v. In this research paper, we will compute the Hosoya polynomial of the Cartesian product of cycles $C_m \times C_n$ for all even numbers m and n being odd $(\forall m \geq n)$.

1. Introduction

Let G be a connected graph, the vertex set and edge set of G is denoted by V(G) and E(G) respectively. The distance d(u,v) between u and v is the length of the smallest path, where u,v∈V(G). The maximum distance between the two vertices of a graph G is called the diameter of G and is denoted by d(G). The degree of a vertex u ∈V(G) is the number of vertices joined to u or the number of edges incident with u and is denoted by d_u. The Hosoya polynomial of a graph G is a generating function that indicates about the distribution of distance in a graph. The polynomial was introduced by a Japanese chemist Haruo Hosoya in 1988. Haruo Hosoya discovered a new formula for the Wiener Index in terms of graph distance and therefore this polynomial is known by the name of its discoverer. The Hosoya polynomial of a connected graph G is defined as (Hosoya, 1988):

$$H(G,x) = \sum_{k=1}^{l} d_k x^k$$

where l is the diameter of G and d_k is the number of paths of length k between the two vertices of G.

In 1996, the Wiener polynomial was solitarily initiated and examined. In fact, the polynomial was originally known as the Wiener polynomial but later, under the admiration of the researcher the name was changed to Hosoya polynomial. The fringe benefit of the Hosoya polynomial is that it contains abundance of knowledge about graph invariants that are distance based. For example, the first derivative of the Hosoya polynomial at x=1 is equal to the Wiener Index. This property of the Hosoya polynomial makes it phenomenal. The Hosoya polynomial gives a supplemental knowledge about distances in a graph G (Sagan *et al*., 1996).

The most interesting application of Hosoya polynomials (Amin *et al*., 2017) is that almost all distance-based graph invariants, which are used to predict physical, chemical, and pharmacological properties of organic molecules can be recovered from Hosoya polynomials. In fact, it calculates the number of distances of paths of different lengths in the graph G (Amin *et al*., 2017).

The Hosoya polynomial of various chemical structures has been determined (Ali & Ali, 2011, Farahani, 2013 and Sadeghieh *et al*., 2017). Moreover, the Hosoya polynomial of some graph families have been examined (Farahani, 2015, Narayankar *et al*., 2012). Also, the Hosoya polynomial of families of graphs has been studied (Stevanovic, 2001 and Wang *et al*., 2016).

The Wiener Index (Rezai *et al*., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$W(G) = H'(G,x)|_{x=1}$$

The hyper Wiener Index (Rezai *et al*., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$WW(G) = H'(G,x)|_{x=1} + \frac{1}{2}H''(G,x)|_{x=1}$$

where the former and later are the first and second derivatives of the Hosoya polynomial at x=1.

2. Materials and Methods

After going through a series of research papers that are based on the Hosoya polynomial of families of graphs, a simple calculation for finding out the Hosoya polynomial, Wiener Index and hyper Wiener Index will be put forward in order to understand these terminologies in a better way.

Create the first form of the understudy family of the graph, and then calculate the distance between the vertices of the graph. After calculating the distance, separate the vertices according to their degrees. Calculate $d(G,k)$ where k denotes the distance parameter. Now, after the completion of this process, revise the same steps for the second form of the family of the graph, then the third, and so on. Once several forms of graphs are checked, a formula is generated that can satisfy the number of paths calculated for each $d(G,k)$. Combining the terms will give the Hosoya polynomial of the family of the graph.

3. Results

In this section, we determine the Hosoya polynomial, Wiener Index and hyper Wiener Index of the families of the Cartesian product of Cycles $C_m \times C_n$, for m, n being even and odd.

First consider the definition of the Cartesian product of Cycles.

3.1 Definition:

The Cartesian product of $C_m \times C_n$ is a graph containing mn vertices and $2mn$ edges, $\forall m, n \geq 3$, where $m \geq n$ and both m and n are odd and even. It is a graph that consists of n cycles and each cycle consists of m vertices joined in such a way that the vertex $u_{1,1}$ of the inner most cycle is connected to the vertex $u_{2,1}$ of the cycle next to the inner most one and $u_{n,1}$ of the exterior most cycle. The vertex $u_{2,1}$ is then connected to the vertex $u_{3,1}$ lying on the third cycle as the index is indicating. Thus, continuing in this manner the vertex $u_{n-1,1}$ is then connected to $u_{n,1}$. The graph $C_m \times C_n$ consists of $m + n$ cycles (Sehar 2014 and Govorcin & Skrekovski 2014).

Theorem 3.1: The Hosoya polynomial of the families of the Cartesian product of Cycles $C_m \times C_n$, where $m \geq n$, m is even and n is odd is

$$H(C_m \times C_n, x) = d(C_m \times C_n, 1)x + d(C_m \times C_n, 2)x^2 + \sum_{r=\frac{n+1}{2}}^{\frac{m-2}{2}} n^2 m x^r + \cdots + d(C_m \times C_n, d)x^d$$

where $\frac{n+1}{2} \leq r \leq \frac{m-2}{2}$ and $d = \frac{m}{2} + \frac{n-1}{2}$ is the diameter of $C_m \times C_n$.

Proof:

Let $G = C_m \times C_n$ be a graph $\forall m, n \geq 3$ with mn vertices and $2mn$ edges. There are vertices of degree 4 only. So, there is no partitioning of the vertices required here. The total number of vertices of degree 4 are mn. The vertex set $V(C_m \times C_n)$ is as follows:

$$V_4 = \{v \in V(C_m \times C_n) | d_v = 4\} \rightarrow |V_4| = mn \quad (3.1.1)$$

Now we know that,

$$|E(G)| = \frac{1}{2} \sum_{k=\delta}^{\Delta} |V_k| \times k \quad (3.1.2)$$

where Δ and δ are the maximum and minimum of $d_v, v \in V(G)$, respectively, thus

$$|E(C_m \times C_n)| = \frac{1}{2}\{4 \times |V_4|\} \quad (3.1.3)$$

Making substitutions from (3.1.1) in (3.1.3),

$$|E(C_m \times C_n)| = \frac{1}{2}\{4mn\} = 2mn \quad (3.1.4)$$

Now to compute the Hosoya polynomial of $C_m \times C_n$, we will use the definition of the Hosoya polynomial from (Hosoya, 1988). Thus, we have

$$H(G,x) = \sum_{k=1}^{d(G)} d(G,k)x^k \qquad (3.1.5)$$

where $d(G,k)$ is the representation of the distance $d(u,v) = k$ and $1 \leq k \leq diam(G)$.

As the diameter of $C_m \times C_n$ ($\forall m,n \geq 3, m \geq n$, m being even and n being odd) is (Sehar, 2014)

$$diam(C_m \times C_n) = \frac{m}{2} + \frac{n-1}{2} = \frac{m+n-1}{2} \qquad (3.1.6)$$

To determine the Hosoya polynomial of $C_m \times C_n$, we will consider the different cases. The technique is that we keep n fixed and will vary m.

Case I: When $m \geq 4$ and $n = 3$

The graph $C_m \times C_3$ have $3m$ vertices and $6m$ edges. Moreover, it is easy to verify that the vertices appearing in the respective families of graphs are of degree 4 and they are $3m$ in numbers. Thus, the vertex set is

$$V_4 = \{v \in V(C_m \times C_3) | d_v = 4\} \rightarrow |V_4| = 3m \qquad (3.1.7)$$

and the total number of edges are

$$|E(C_m \times C_3)| = \frac{1}{2}\{4 \times |V_4|\}$$

$$|E(C_m \times C_3)| = \frac{1}{2}\{12m\} = 6m \qquad (3.1.8)$$

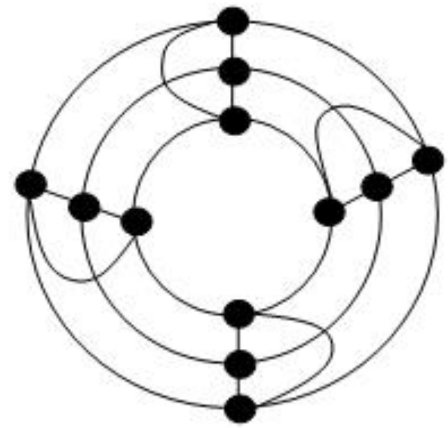

Figure 1: $C_4 \times C_3$

As, the diameter of $C_m \times C_n$ is $d = \frac{m}{2} + \frac{n-1}{2}$, so for $C_m \times C_3$ it is $\frac{m+3-1}{2} = \frac{m+2}{2}$. It is clear from the definition of the edge set of $C_m \times C_3$, that the number of 1-edge path is $6m$. Hence,

$$d(C_m \times C_3, 1) = |E(C_m \times C_3)| = 6m \qquad (3.1.9)$$

$$d(C_m \times C_3, r) = 9m, 2 \leq r \leq \frac{m-2}{2} \qquad (3.1.10)$$

There are $9m$ r-edges paths between $u, v \in V_4$, where $2 \leq r \leq \frac{m-2}{2}$. Hence, we get the second term which is of the form $[9m]x^r$.

$$d\left(C_m \times C_3, \frac{m}{2}\right) = \frac{15m}{2} \qquad (3.1.11)$$

The number of $\frac{m}{2}$-edges paths between the vertices $u, v \in V_4$ are $\frac{15m}{2}$. Thus, the third term of the Hosoya polynomial is of the form $[\frac{15m}{2}]x^{\frac{m}{2}}$.

$$d\left(C_m \times C_3, \frac{m+2}{2}\right) = 3m \qquad (3.1.12)$$

The number of $\frac{m+2}{2}$-edges paths between the vertices $u, v \in V_4$ are $3m$. Thus, the last term of the Hosoya polynomial is of the form $[3m]x^{\frac{m+2}{2}}$.

Now, adding up all the distances, we get the following form of the Hosoya polynomial of $C_m \times C_3$,

$$H(C_m \times C_3) = 6mx + \sum_{r=2}^{\frac{m-1}{2}} 9mx^r + \frac{15m}{2}x^{\frac{m}{2}} + 3mx^{\frac{m+2}{2}}$$

This completes the Case I.

Case II: When $m \geq 6$ and $n = 5$

The graph $C_m \times C_5$ have $5m$ vertices and $10m$ edges. Furthermore, one can make a note of that the only vertices that appear in the under-study family is of degree 4. So, the total number of vertices of degree 4 are $5m$. Hence, the vertex set is

$$V_4 = \{v \in V(C_m \times C_5) | d_v = 4\} \rightarrow |V_4| = 5m \qquad (3.1.13)$$

and the total number of edges are

$$|E(C_m \times C_5)| = \frac{1}{2}\{20m\} = 10m \qquad (3.1.14)$$

The diameter of $C_m \times C_5$ is $\frac{m+n-1}{2} = \frac{m+5-1}{2} = \frac{m+4}{2}$. From the definition and structure of the respective family, it is easy to see that the number of 1-edge path is equal to the total number of edges. Hence,

$$d(C_m \times C_5, 1) = |E(C_m \times C_5)| = 10m \qquad (3.1.15)$$

$$d(C_m \times C_5, 2) = 20m \qquad (3.1.16)$$

The number of 2-edges paths between the vertices $u, v \in V_4$ are 20m. Thus, the second sentence term of the Hosoya polynomial is of the form $[20m]x^2$.

$$d(C_m \times C_5, r) = 25m, 3 \leq r \leq \frac{m-2}{2} \qquad (3.1.17)$$

The number of r-edges paths between $u, v \in V_4$ are $25m$, where $3 \leq r \leq \frac{m-2}{2}$. Thus, we get the term $[25m]x^r$.

$$d\left(C_m \times C_5, \frac{m}{2}\right) = \frac{45m}{2} \qquad (3.1.18)$$

The number of $\frac{m}{2}$-edges paths between the vertices $u, v \in V_4$ are $\frac{45m}{2}$. Thus, for the corresponding $\frac{m}{2}$ term of the polynomial we have, $[\frac{45m}{2}]x^{\frac{m}{2}}$.

$$d\left(C_m \times C_5, \frac{m+2}{2}\right) = 15m \qquad (3.1.19)$$

The number of $\frac{m+2}{2}$-edges paths between the vertices $u, v \in V_4$ are $15m$. Thus, the fifth term of the polynomial is, $[15m]x^{\frac{m+2}{2}}$.

$$d\left(C_m \times C_5, \frac{m+4}{2}\right) = 5m \qquad (3.1.20)$$

The number of $\frac{m+4}{2}$-edges paths between the vertices $u, v \in V_4$ are $5m$. Thus, the last term of the polynomial is, $[5m]x^{\frac{m+4}{2}}$.

Adding up all the above calculated distances, we have the following form of the Hosoya polynomial of $C_m \times C_5$,

$$H(C_m \times C_5) = 10mx + 20mx^2 + \sum_{r=3}^{\frac{m-2}{2}} 25mx^r + \frac{45m}{2}x^{\frac{m}{2}} + 15mx^{\frac{m+2}{2}} + 5mx^{\frac{m+4}{2}}$$

This completes the second case. Now, one can easily distinguish the difference between the Hosoya polynomial of $C_m \times C_3$ and $C_m \times C_5$. In the later, there is specific number of 2-edges paths which were not appearing in the former. To be more crystal clear regarding the pattern of the k-edges paths where $1 \le k \le \frac{m+n-1}{2}$. We will consider a third case to come to a conclusion.

Case III: When $m \ge 8$ and $n = 7$

The graph $C_m \times C_7$ have $7m$ vertices and $14m$ edges. Moreover, it is easy to verify that the vertices appearing in the respective family is of degree 4. So, the total number of vertices of degree 4 are $7m$. Hence, the vertex set is

$$V_4 = \{v \in V(C_m \times C_7) | d_v = 4\} \rightarrow |V_4| = 7m \quad (3.1.21)$$

and the total number of edges are

$$|E(C_m \times C_7)| = \frac{1}{2}\{28m\} = 14m \quad (3.1.22)$$

The diameter of $C_m \times C_7$ is $\frac{m+n-1}{2} = \frac{m+7-1}{2} = \frac{m+6}{2}$. It is easy to verify that the number of 1-edge path is equal to the total number of edges. Hence,

$$d(C_m \times C_7, 1) = |E(C_m \times C_7)| = 14m \quad (3.1.23)$$

$$d(C_m \times C_7, 2) = 28m \quad (3.1.24)$$

The number of 2-edges paths between the vertices $u, v \in V_4$ are $28m$. Thus, the second sentence term of the Hosoya polynomial is of the form $[28m]x^2$.

$$d(C_m \times C_7, 3) = 42m \quad (3.1.25)$$

The number of 3-edges paths between the vertices $u, v \in V_4$ are $42m$. Thus, the third sentence term of the Hosoya polynomial is of the form $[42m]x^3$.

$$d(C_m \times C_7, r) = 49m, 4 \le r \le \frac{m-2}{2} \quad (3.1.26)$$

The number of r-edges paths between $u, v \in V_4$ are $49m$, where $4 \le r \le \frac{m-2}{2}$. Thus, we get the term $[49m]x^r$.

$$d\left(C_m \times C_7, \frac{m}{2}\right) = \frac{91m}{2} \quad (3.1.27)$$

The number of $\frac{m}{2}$-edges paths between the vertices $u, v \in V_4$ are $\frac{91m}{2}$. Thus, for the corresponding $\frac{m}{2}$ term of the polynomial we have, $[\frac{91m}{2}]x^{\frac{m}{2}}$.

$$d\left(C_m \times C_7, \frac{m+2}{2}\right) = 35m \qquad (3.1.28)$$

The number of $\frac{m+2}{2}$-edges paths between the vertices $u, v \in V_4$ are $35m$. Thus, the third last term of the polynomial is, $[35m]x^{\frac{m+2}{2}}$.

$$d\left(C_m \times C_7, \frac{m+4}{2}\right) = 21m \qquad (3.1.29)$$

The number of $\frac{m+4}{2}$-edges paths between the vertices $u, v \in V_4$ are $21m$. Thus, the second last term of the polynomial is, $[21m]x^{\frac{m+4}{2}}$.

$$d\left(C_m \times C_7, \frac{m+6}{2}\right) = 7m \qquad (3.1.30)$$

The number of $\frac{m+6}{2}$-edges paths between the vertices $u, v \in V_4$ are $7m$. Thus, the last term of the polynomial is, $[7m]x^{\frac{m+6}{2}}$.

Adding up all the above determined distances, we have the following form of the Hosoya polynomial of $C_m \times C_7$,

$$H(C_m \times C_7) = 14mx + +28mx^2 + 42mx^3 + \sum_{r=4}^{\frac{m-2}{2}} 49mx^r + \frac{91m}{2}x^{\frac{m}{2}} + 35mx^{\frac{m+2}{2}} + 21mx^{\frac{m+4}{2}} + 7mx^{\frac{m+6}{2}}$$

Thus, we acquire the desired result after keeping in view the pattern of the distance distribution among the vertices of every graph.

Hence, we get the desired Hosoya polynomial for this family of graph i.e.

$$H(C_m \times C_n, x) = d(C_m \times C_n, 1)x + d(C_m \times C_n, 2)x^2 + \sum_{r=\frac{n+1}{2}}^{\frac{m-2}{2}} n^2mx^r + \cdots + d(C_m \times C_n, d)x^d$$

where $\frac{n+1}{2} \leq r \leq \frac{m-2}{2}$ and $d = \frac{m}{2} + \frac{n-1}{2}$is the diameter of $C_m \times C_n$.

This completes the proof.

4. Discussion

In this paper, we have determined the Hosoya polynomial of the Cartesian product of cycles $C_m \times C_n$ for m being even and n being odd $(\forall m \geq n)$.

5. References

Ali, A. & Ali, A. M. (2011). Hosoya Polynomial of Pentachains. *Match Communications in. Mathematical and in. Computer. Chemistry*. 65, 807-819.

Amin, N., Nizami, A. R., & Idrees, M. (2017). Some Topological Invariants of Generalized Mobius Ladder. [math.CO].

Farahani, M. R. (2015). The Wiener Index and Hosoya Polynomial of a class of Jahangir graphs $J_{3,m}$. Fundamental *Journal of Mathematics and Mathematical Sciences*. 3(1), 91-96.

Farahani, M. R. (2015). Hosoya Polynomial of Jahangir Graphs $J_{4,m}$. *Global Journal of Mathematics*. 3(1), 232-236.

Farahani, M. R. (2013). On the Schultz polynomial and Hosoya polynomial of Circumcoronene Series of Benzenoid. *Journal of. Applied Mathematics. and Informatics*. 31(5-6), 595-608.

Govorcin, J. & Skrekovski, R. (2014). On the connectivity of Cartesian product of graphs. *ARS Mathematica contemporanea*. 7, 293-297.

Hosoya, H. (1988). On some counting polynomials in chemistry. *Discrete Applied Mathematics*. 19, 239-257.

Narayankar, K. P., B., L. S., Mathad, V. & Gutman, I. (2012). Hosoya polynomial of Hanoi Graphs. Kragujevac *Journal of Mathematics*. 36(1), 51-57.

Rezai, M., Farahani, M. R., Khalid, W. & Baig, A. Q. (2017). Computation of Hosoya Polynomial, Wiener and Hyper Wiener Index of Jahangir Graph $J_{6,m}$. *Journal of Prime Research in Mathematics*. 13, 30-40.

Sadeghieh, A., Alikhani, S., Ghanbari, N. & Khalaf, A. J. M. (2017). Hosoya polynomial of some cactus chains. *Cogent Mathematics*. 4.

Sagan, B. E., Yeh, Y. N. & Zhang, P. (1996). The Wiener Polynomial of a graph. *International Journal of Quantum Chemistry*. 60, 959-969.

Sehar, A. (2014). The center and periphery for specific families of graphs. Unpublished MS. Thesis, Lahore College for Women University, Pakistan.

Stevanovic, D. (2011). Hosoya Polynomial of composite graphs. *Discrete Mathematic*s. 235, 237-244.

Wang, S., Farahani, M. R., Kanna, M. R. R., Jamil, M. K. & Kumar, R. P. (2016). The Wiener Index and the Hosoya Polynomial of the Jahangir Graphs. *Applied and Computational Mathematics*. 5(3), 138-141.