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HOSOYA POLYNOMIAL OF CARTESIAN PRODUCT OF CYCLES $C_m \times C_n$, $m \geq n$, m & n BEING EVEN

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Abstract

Let G be a simple connected graph having vertex set $V(G)$ and edge set $E(G)$. The Hosoya polynomial of G is $H(G,x) = \sum_{\{u,v\} \subset V(G)} x^{d(u,v)}$, where $d(u,v)$ denotes the distance between the vertices u and v. In this research paper, we will compute the Hosoya polynomial of the Cartesian product of cycles $C_m \times C_n$ for all even numbers and where $m \geq n$.

1. Introduction

Let G be a connected graph, the vertex set and edge set of G is denoted by $V(G)$ and $E(G)$ respectively. The distance $d(u,v)$ between u and v is the length of the smallest path, where $u, v \in V(G)$. The maximum distance between the two vertices of a graph G is called the diameter of G and is denoted by $d(G)$. The degree of a vertex $u \in V(G)$ is the number of vertices joined to u or the number of edges incident with u and is denoted by d_u. The Hosoya polynomial of a graph G is a generating function that indicates about the distribution of distance in a graph. The polynomial was introduced by a Japanese chemist Haruo Hosoya in 1988. Haruo Hosoya discovered a new formula for the Wiener Index in terms of graph distance and therefore this polynomial is known by the name of its discoverer. The Hosoya polynomial of a connected graph G is defined as (Hosoya, 1988):

$$H(G,x) = \frac{1}{2} \sum_{v \in V(G)} V(G) \sum_{u \in V(G)} V(G)\ d(u,v)$$

The Hosoya polynomial of various chemical structures has been determined (Ali & Ali, 2011, Farahani, 2013 and Sadeghieh et al., 2017). Moreover, the Hosoya polynomial of some graph families have been examined (Farahani, 2015, Farahani, 2015, Narayankar et al., 2012). Also, the Hosoya polynomial of families of graphs has been studied (Stevanovic, 2001 and Wang et al., 2016).

The Wiener Index (Rezai et al., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$W(G) = \frac{\partial H(G,x)}{\partial x}|_{x=1}$$

The hyper Wiener Index (Rezai et al., 2017) of a graph can be calculated by using the Hosoya polynomial. It is formulated as follows:

$$WW(G) = H'(G,x)|_{x=1} + \frac{1}{2}H''(G,x)|_{x=1}$$

where the former and later are the first and second derivatives of the Hosoya polynomial at $x = 1$.

1.1 Definition

The Cartesian product of $C_m \times C_n$ is a graph containing mn vertices and $2mn$ edges, $\forall m, n \geq 3$, where $m \geq n$ and both m and n are odd and even. It is a graph that consists of n cycles and each cycle consists of m vertices joined in such a way that the vertex $u_{1,1}$ of the inner most cycle is connected to the vertex $u_{2,1}$ of the cycle next to the inner most one and $u_{n,1}$ of the exterior most cycle. The vertex $u_{2,1}$ is then connected to the vertex $u_{3,1}$ lying on the third cycle as the index is indicating. Thus, continuing in this manner the vertex $u_{n-1,1}$ is then connected to $u_{n,1}$. The graph $C_m \times C_n$ consists of $m + n$ cycles (Sehar 2014 and Govorcin & Skrekovski 2014).

2. Materials and Methods

A simple calculation for finding out the Hosoya polynomial, Wiener Index and hyper Wiener Index will be put forward in order to understand these.

3. Results

In this section, we determine the Hosoya polynomial, Wiener Index and hyper Wiener Index of the families of the Cartesian product of Cycles $C_m \times C_n$, for m, n both even.

Theorem 3.1: The Hosoya polynomial of the families of the Cartesian product of Cycles $C_m \times C_n$, where $m > n$ and both m and n are even is

$$H(C_m \times C_n, x) = d(C_m \times C_n, 1)x + d(C_m \times C_n, 2)x^2 + \sum_{r=\frac{n+2}{2}}^{\frac{m-2}{2}} n^2 m x^r + \cdots + d(C_m \times C_n, d)x^d$$

where $\frac{n+2}{2} \leq r \leq \frac{m-2}{2}$ and $d = \frac{m+n}{2}$ is the diameter of $C_m \times C_n$.

Proof

Let $G = C_m \times C_n$ be a graph $\forall m, n \geq 4$ with mn vertices and $2mn$ edges. There are vertices of degree 4 only. So, there is no partitioning of the vertices required here. The total number of vertices of degree 4 are mn. The vertex set

$V(C_m \times C_n)$ is as follows:

$$V_4 = \{v \in V(C_m \times C_n) | d_v = 4\} \to |V_4| = mn \quad (3.1.1)$$

Now we know that,

$$|E(G)| = \frac{1}{2}\sum_{k=\delta}^{\Delta} |V_k| \times k \quad (3.1.2)$$

where Δ and δ are the maximum and minimum of $d_v, v \in V(G)$, respectively, thus

$$|E(C_m \times C_n)| = \frac{1}{2}\{4 \times |V_4|\} \quad (3.1.3)$$

Making substitutions from (3.1.1) in (3.1.3),

$$|E(C_m \times C_n)| = \frac{1}{2}\{4mn\} = 2mn \quad (3.1.4)$$

Now to compute the Hosoya polynomial of $C_m \times C_n$, we will use the definition of the Hosoya polynomial from (Hosoya, 1988). Thus, we have

$$H(G,x) = \sum_{k=1}^{d(G)} d(G,k)x^k \quad (3.1.5)$$

where $d(G,k)$ is the representation of the distance $d(u,v) = k$ and $1 \leq k \leq diam(G)$.

As the diameter of $C_m \times C_n$ ($\forall m,n \geq 3, m \geq n$ and both m,n are even) is (Sehar, 2014)

$$diam(C_m \times C_n) = \frac{m+n}{2} \quad (3.1.6)$$

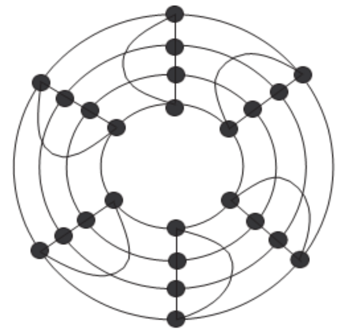

Figure 1: $C_6 \times C_4$

To determine the Hosoya polynomial of $C_m \times C_n$, we will consider the different cases. The strategy is that we keep n fixed and will vary m in order to obtain the Hosoya polynomial for the case when $m > n$.

Case I: When $m \geq 6$ and $n = 4$

The graph $C_m \times C_4$ have $4m$ vertices and $8m$ edges. Moreover, it is easy to verify that the vertices appearing in the respective families of graphs are of degree 4 and they are $4m$ in numbers. Thus, the vertex set is

$$V_4 = \{v \in V(C_m \times C_4) | d_v = 4\} \to |V_4| = 4m \quad (3.1.7)$$

and the total number of edges are

$$|E(C_m \times C_4)| = \frac{1}{2}\{4 \times |V_4|\}$$

$$|E(C_m \times C_4)| = \frac{1}{2}\{16m\} = 8m \quad (3.1.8)$$

The diameter of $C_m \times C_4$ it is $\frac{m+4}{2}$. It is clear from the definition of the edge set of $C_m \times C_4$, that the number of 1-edge path is $8m$. Hence,

$$d(C_m \times C_4, 1) = |E(C_m \times C_4)| = 8m \quad (3.1.9)$$

$$d(C_m \times C_4, 2) = 14m \quad (3.1.10)$$

The number of 2-edges paths between the vertices of $u, v \in V_4$ are $14m$. Thus, the second term of the Hosoya polynomial is of the form $[14m]x^2$.

$$d(C_m \times C_4, r) = 16m, 3 \leq r \leq \frac{m-2}{2} \quad (3.1.11)$$

There are $16m$ r-edges paths between $u, v \in V_4$, where $3 \leq r \leq \frac{m-2}{2}$. Hence, we get the third term which is of the form $[16m]x^r$.

$$d\left(C_m \times C_4, \frac{m}{2}\right) = 14m \quad (3.1.12)$$

The number of $\frac{m}{2}$-edges paths between the vertices $u, v \in V_4$ are $14m$. Thus, the fourth term of the Hosoya polynomial is of the form $[14m]x^{\frac{m}{2}}$.

$$d\left(C_m \times C_4, \frac{m+2}{2}\right) = 8m \quad (3.1.13)$$

The number of $\frac{m+2}{2}$-edges paths between the vertices $u, v \in V_4$ are $8m$. Thus, the fifth term of the Hosoya polynomial is of the form $[8m]x^{\frac{m+2}{2}}$.

$$d\left(C_m \times C_4, \frac{m+4}{2}\right) = 2m \quad (3.1.14)$$

The number of $\frac{m+4}{2}$-edges paths between the vertices $u, v \in V_4$ are $2m$. Thus, the last term of the Hosoya polynomial is of the form $[2m]x^{\frac{m+4}{2}}$.

Now, adding up all the distances, we get the following form of the Hosoya polynomial of $C_m \times C_4$,

$$H(C_m \times C_4, x) = 8mx + 14mx^2 + \sum_{r=3}^{\frac{m-2}{2}} 16mx^r + 14mx^{\frac{m}{2}} + 8mx^{\frac{m+2}{2}} + 2mx^{\frac{m+4}{2}}$$

This completes the Case I.

Case II: When $m \geq 8$ and $n = 6$

The graph $C_m \times C_6$ have $6m$ vertices and $12m$ edges. Furthermore, one can make a note of that the only vertices that appear in the under-study family is of degree 4. So, the total number of vertices of degree 4 are $6m$. The total number of edges are

$$|E(C_m \times C_6)| = \frac{1}{2}\{24m\} = 12m \quad (3.1.15)$$

The diameter of $C_m \times C_6$ is $\frac{m+6}{2}$. From the definition and structure of the respective family, it is easy to see that the number of 1-edge path is equal to the total number of edges. Hence,

$$d(C_m \times C_6, 1) = |E(C_m \times C_6)| = 12m \quad (3.1.16)$$

$$d(C_m \times C_6, 2) = 24m \quad (3.1.17)$$

The number of 2-edges paths between the vertices $u, v \in V_4$ are $24m$. Thus, the second term of the Hosoya polynomial is of the form $[24m]x^2$.

$$d(C_m \times C_6, 3) = 33m \quad (3.1.18)$$

The number of 3-edges paths between the vertices $u, v \in V_4$ are $33m$. Thus, the third term of the Hosoya polynomial is of the form $[33m]x^3$.

$$d(C_m \times C_6, r) = 36m, 4 \leq r \leq \frac{m-2}{2} \quad (3.1.19)$$

The number of r-edges paths between $u, v \in V_4$ are $36m$, where $4 \leq r \leq \frac{m-2}{2}$. Thus, we get the term $[36m]x^r$.

$$d\left(C_m \times C_6, \frac{m}{2}\right) = 33m \quad (3.1.20)$$

The number of $\frac{m}{2}$-edges paths between the vertices $u, v \in V_4$ are $33m$. Thus, for the corresponding $\frac{m}{2}$ term of the polynomial we have, $[33m]x^{\frac{m}{2}}$.

$$d\left(C_m \times C_6, \frac{m+2}{2}\right) = 24m \quad (3.1.21)$$

The number of $\frac{m+2}{2}$-edges paths between the vertices $u, v \in V_4$ are $24m$. Thus, for this corresponding term of the polynomial we have, $[24m]x^{\frac{m+2}{2}}$.

$$d\left(C_m \times C_6, \frac{m+4}{2}\right) = 12m \quad (3.1.22)$$

The number of $\frac{m+4}{2}$-edges paths between the vertices $u, v \in V_4$ are $12m$. Thus, for this corresponding term of the polynomial we have, $[12m]x^{\frac{m+4}{2}}$.

$$d\left(C_m \times C_6, \frac{m+6}{2}\right) = 3m \quad (3.1.23)$$

The number of $\frac{m+6}{2}$-edges paths between the vertices $u, v \in V_4$ are $3m$. Thus, for last term of the polynomial we have, $[3m]x^{\frac{m+6}{2}}$.

Adding up all the above calculated distances, we have the following form of the Hosoya polynomial of $C_m \times C_6$,

$$H(C_m \times C_6, x) = 12mx + 24mx^2 + 33mx^3 + \sum_{r=4}^{\frac{m-2}{2}} 36mx^r + 33mx^{\frac{m}{2}} + 24mx^{\frac{m+2}{2}} + 12mx^{\frac{m+4}{2}} + 3mx^{\frac{m+6}{2}}$$

This completes the second case. Now, one can easily distinguish the difference between the Hosoya polynomial of $C_m \times C_4$ and $C_m \times C_6$. In the later, there is specific number of 2-edges paths which were not appearing in the former. To be more crystal clear regarding the pattern of the k-edges paths where $1 \leq k \leq \frac{m+n}{2}$. We will consider a third case to come to a conclusion.

Case III: When $m \geq 10$ and $n = 8$

The graph $C_m \times C_8$ have $8m$ vertices and $16m$ edges. Moreover, it is easy to verify that the vertices appearing in the respective family is of degree 4. So, the total number of vertices of degree 4 are $8m$. The total number of edges are

$$|E(C_m \times C_8)| = \frac{1}{2}\{32m\} = 16m \quad (3.1.25)$$

The diameter of $C_m \times C_8$ is $\frac{m+8}{2}$. It is easy to verify that the number of 1-edge path is equal to the total number of edges. Hence,

$$d(C_m \times C_8, 1) = |E(C_m \times C_8)| = 16m \quad (3.1.26)$$

$$d(C_m \times C_8, 2) = 32m \quad (3.1.27)$$

The number of 2-edges paths between the vertices $u, v \in V_4$ are $32m$. Thus, the second sentence of the Hosoya polynomial is of the form $[32m]x^2$.

$$d(C_m \times C_8, 3) = 48m \quad (3.1.28)$$

The number of 3-edges paths between the vertices $u, v \in V_4$ are $48m$. Thus, the third term of the Hosoya polynomial is of the form $[48m]x^3$.

$$d(C_m \times C_8, 4) = 60m \quad (3.1.29)$$

The number of 4-edges paths between the vertices $u, v \in V_4$ are $60m$. Thus, the fourth term of the Hosoya polynomial is of the form $[60m]x^4$.

$$d(C_m \times C_8, r) = 64m, 5 \le r \le \frac{m-2}{2} \quad (3.1.30)$$

The number of r-edges paths between $u, v \in V_4$ are $64m$, where $5 \le r \le \frac{m-2}{2}$. Thus, we get the term $[64m]x^r$.

$$d\left(C_m \times C_8, \frac{m}{2}\right) = 60m \quad (3.1.31)$$

The number of $\frac{m}{2}$-edges paths between the vertices $u, v \in V_4$ are $60m$. Thus, for the corresponding $\frac{m}{2}$ term of the polynomial we have, $[60m]x^{\frac{m}{2}}$.

$$d\left(C_m \times C_8, \frac{m+2}{2}\right) = 48m \quad (3.1.32)$$

The number of $\frac{m+2}{2}$-edges paths between the vertices $u, v \in V_4$ are $48m$. Thus, for the corresponding $\frac{m+2}{2}$ term of the polynomial we have, $[48m]x^{\frac{m+2}{2}}$.

$$d\left(C_m \times C_8, \frac{m+4}{2}\right) = 32m \quad (3.1.33)$$

The number of $\frac{m+4}{2}$-edges paths between the vertices $u, v \in V_4$ are $32m$. Thus, for the corresponding $\frac{m+4}{2}$ term of the polynomial we have, $[32m]x^{\frac{m+4}{2}}$.

$$d\left(C_m \times C_8, \frac{m+6}{2}\right) = 16m \quad (3.1.34)$$

The number of $\frac{m+6}{2}$-edges paths between the vertices $u, v \in V_4$ are $16m$. Thus, for the corresponding $\frac{m+6}{2}$ term of the polynomial we have, $[16m]x^{\frac{m+6}{2}}$.

$$d\left(C_m \times C_8, \frac{m+8}{2}\right) = 4m \quad (3.1.35)$$

The number of $\frac{m+8}{2}$-edges paths between the vertices $u, v \in V_4$ are $4m$. Thus, the last term of the polynomial is, $[4m]x^{\frac{m+8}{2}}$.

Adding up all the above determined distances, we have the following form of the Hosoya polynomial of $C_m \times C_8$,

$$H(C_m \times C_8, x) = 16mx + 32mx^2 + 48mx^3 + 60mx^4 + \sum_{r=5}^{\frac{m-2}{2}} 64mx^r + 60mx^{\frac{m}{2}} + 48mx^{\frac{m+2}{2}} + 32mx^{\frac{m+4}{2}} + 16mx^{\frac{m+6}{2}} + 4mx^{\frac{m+8}{2}}$$

Thus, we acquire the desired result after keeping in view the pattern of the distance distribution among the vertices of every graph.

This completes the proof.

Theorem 3.2: The Hosoya polynomial of the families of the Cartesian product of Cycles $C_m \times C_n$, where $m = n$ and both m and n are even is

$$H(C_m \times C_m, x) = d(C_m \times C_m, 1)x + d(C_m \times C_m, 2)x^2 + \sum_{r=\frac{m}{2}}^{\frac{m}{2}} (m^3 - m^2)x^r + \cdots + d(C_m \times C_m, d)x^d$$

Proof

There are total mn vertices and $2mn$ edges. To compute the Hosoya polynomial for the case when $m = n$, we will use the same methodology as used in the above Theorem. We will consider three cases and come to the conclusion. The diameter (Sehar, 2014) of $C_m \times C_n$ when both m, n are even and $m = n$ is,

$$diam(C_m \times C_m) = \frac{m+m}{2} = m \qquad (3.2.1)$$

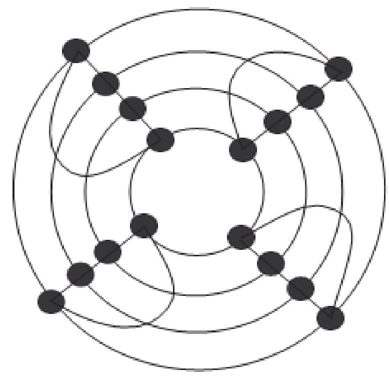

Figure 2: $C_4 \times C_4$

By using the definition of the Hosoya polynomial from (Hosoya, 1988), we will proceed as follows:

Case I: When $m = 4$ and $n = 4$

$$d(C_4 \times C_4, 1) = 32 \qquad (3.2.2)$$

$$d(C_4 \times C_4, 2) = 48 \qquad (3.2.3)$$

The number of 2-edges paths between the vertices of $u, v \in V_4$ are 48. Thus, the second term of the Hosoya polynomial is of the form $48x^2$.

$$d(C_4 \times C_4, 3) = 32 \qquad (3.2.4)$$

There are 32, 3-edges paths between $u, v \in V_4$. Hence, we get the third term which is of the form $32x^3$.

$$d(C_4 \times C_4, 4) = 8 \qquad (3.2.5)$$

The number of 4-edges paths between the vertices $u, v \in V_4$ are 8. Thus, the last term of the Hosoya polynomial is of the form $8x^4$.

Now, adding up all the distances, we get the following form of the Hosoya polynomial of $C_m \times C_4$,

$$H(C_4 \times C_4, x) = 32x + 48x^2 + 32x^3 + 8x^4$$

This completes the Case I.

Case II: When $m = 6$ and $n = 6$

From the definition and structure of the respective family, it is easy to see that the number of 1-edge path is equal to the total number of edges. Hence,

$$d(C_6 \times C_6, 1) = 72 \quad (3.2.6)$$

$$d(C_6 \times C_6, 2) = 144 \quad (3.2.7)$$

The number of 2-edges paths between the vertices $u, v \in V_4$ are 144. Thus, the second term of the Hosoya polynomial is of the form $144x^2$.

$$d(C_6 \times C_6, 3) = 180 \quad (3.2.8)$$

The number of 3-edges paths between the vertices $u, v \in V_4$ are 180. Thus, the third term of the Hosoya polynomial is of the form $180x^3$.

$$d(C_6 \times C_6, 4) = 144 \quad (3.2.9)$$

The number of 4-edges paths between $u, v \in V_4$ are 144. Thus, we get the term $144x^4$.

$$d(C_6 \times C_6, 5) = 72 \quad (3.2.10)$$

The number of 5-edges paths between the vertices $u, v \in V_4$ are 72. Thus, we have, $72x^5$.

$$d(C_6 \times C_6, 6) = 18 \quad (3.2.11)$$

The number of 6-edges paths between the vertices $u, v \in V_4$ are 18. Thus, for the last term of the polynomial we have, $18x^6$.

Hence the Hosoya polynomial for $C_6 \times C_6$,

$$H(C_6 \times C_6) = 72x + 144x^2 + 180x^3 + 144x^4 + 72x^5 + 18x^6$$

This completes the second case. To come to a conclusion regarding this result, we will consider a third and last case.

Case III: When $m = 8$ and $n = 8$

For the first sentence of the Hosoya polynomial, we have

$$d(C_8 \times C_8, 1) = 128 \quad (3.2.12)$$

$$d(C_8 \times C_8, 2) = 256 \quad (3.2.13)$$

The number of 2-edges paths between the vertices $u, v \in V_4$ are 256. Thus, the second sentence of the Hosoya polynomial is of the form $256x^2$.

$$d(C_8 \times C_8, 3) = 384 \quad (3.2.14)$$

The number of 3-edges paths between the vertices $u, v \in V_4$ are 384. Thus, the third term of the Hosoya polynomial is of the form $384x^3$.

$$d(C_8 \times C_8, 4) = 448 \quad (3.2.15)$$

The number of 4-edges paths between the vertices $u, v \in V_4$ are 448. Thus, the fourth term of the Hosoya polynomial is of the form $448x^4$.

$$d(C_8 \times C_8, 5) = 384 \quad (3.2.16)$$

The number of 5-edges paths between $u, v \in V_4$ are 384. Thus, the fifth term is $384x^5$.

$$d(C_8 \times C_8, 6) = 256 \quad (3.2.17)$$

The number of 6-edges paths between the vertices $u, v \in V_4$ are 256. Thus, for the corresponding sixth term of the polynomial we have, $256x^6$.

$$d(C_8 \times C_8, 7) = 128 \quad (3.2.18)$$

The number of 7-edges paths between the vertices $u, v \in V_4$ are 128. Thus, for the corresponding seventh term of the polynomial we have, $128x^7$.

$$d(C_8 \times C_8, 8) = 32 \quad (3.2.19)$$

The number of 8-edges paths between the vertices $u, v \in V_4$ are 32. Thus, for the last term of the polynomial we have, $32x^8$.

Adding up all the above determined distances, we have the following form of the Hosoya polynomial of $C_8 \times C_8$,

$$H(C_8 \times C_8, x) = 128x + 256x^2 + 384x^3 + 448x^4 + 384x^5 + 256x^6 + 128x^7 + 32x^8$$

Thus, we acquire the desired result after keeping in view the pattern of the distance distribution among the vertices of every graph.

This completes the proof.

4. Discussion

In this paper, we have determined the Hosoya polynomial of the Cartesian product of cycles $C_m \times C_n$ for all even numbers and where $m \geq n$.

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