



DISCRETE AND INDISCRETE (4, 2)-FUZZY TOPOLOGICAL STRUCTURES

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Abstract

In this paper, we study the concept of (4,2)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Hausdorff space. We also obtain the characteristic of the homomorphic image and inverse image of (4,2)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras). 2020 AMS classification: 03G25, 06F35, 08A72.



1. Introduction

In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition: an element either belongs or does not belong to the set. As an extension, fuzzy set theory (L. A. Zadeh, 1965) permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0; 1]$. As a generalization of fuzzy set, K. T. Atanassov created intuitionistic fuzzy set. Intuitionistic fuzzy set is widely used in all fields (I. Cristea *et al.*, 2010, B. Davvaz *et al.*, 2006, Y. B. Jun & K. H. Kim, 2000, S. Yamak *et al.*, 2008) for applications in algebraic

structures). In 2013, Yager (R. R. Yager, 2013, R. R. Yager, 2013, R. R. Yager & A. M. Abbasov, 2013) introduced Pythagorean fuzzy set and compared it with intuitionistic fuzzy set. Pythagorean fuzzy set is a new extension of intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (S. Bhunia *et al.*, 2020), UP-algebras (A. Satirad *et al.*, 2021) and topological spaces (A. Satirad *et al.*, 2019). Senapati (T. Senapati & R. R. Yager., 2020) introduced Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (I.

Silambarasan, 2021). H. Z. Ibrahim *et al.*, 2021 introduced (3; 2)-fuzzy sets and applied it to topological spaces. In this paper, we study the concept of (4,2)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Hausdorff space. We also obtain the characteristic of the homomorphic image and inverse image of (4,2)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

1.1 Preliminaries and Various Basic Concept of BCC-algebras (BCK-algebras)

In this section, we first review some definitions and properties which will be used in the sequel. A non-empty set G with a constant 0 and binary operation * is called a BCC-algebra if it satisfies the following conditions:

- a) $((x * y) * (z * y)) * (x * y) = 0$
- b) $x * x = 0$
- c) $0 * x = 0$
- d) $x * 0 = 0$
- e) $x * y = 0, y * x = 0 \Rightarrow x = y$

for all $x, y, z \in G$. In BCC-algebra, the following equality holds $(x * y) * x = 0$. Obviously, any BCK-algebra is BCC-algebra but there exist BCC-algebras which are not necessarily BCK-algebra. We note that a BCC-algebra is BCK-algebra if and if only, it satisfies the equality $(x * y) * z = (x * z) * y$. A non-empty subset 'S' of a BCK-algebra 'G' is called a sub algebra of G if it is closed under the BCC-operation. Such algebra contains the constant 0 and it is clearly a BCC-algebra, but some sub algebras may be also BCK-algebras. Moreover, there exit BCC-algebras which all sub algebras are BCK-

algebras. A mapping $\varphi: G_1 \rightarrow G_2$ of BCC-algebras is called a homomorphism if

$$\varphi(x * y) = \varphi(x) * \varphi(y) \text{ holds, for all } x, y \in G_1.$$

For a non-empty given set G, let I be the closed unit interval [0, 1]. Then, an (4,2)-fuzzy set is an object of the form $A = \{ \langle x, \delta_A^4(x), \lambda_A^2(x) \rangle / x \in G \}$, when the mappings $\delta_A^4: G \rightarrow I$ and $\lambda_A^2: G \rightarrow I$ denote the degree of membership (namely, $\delta_A(x)$) and the degree of non-membership (namely, $\lambda_A(x)$) of each element $x \in G$ to the object 'A' respectively satisfying $0 \leq \delta_A^m(x) + \lambda_A^n(x) \leq 1$ for all $x \in G$. The complement of the (4,2)-fuzzy set is $A^c = \{ \langle x, \lambda_A^2(x), \delta_A^4(x) \rangle / x \in G \}$. Obviously, every fuzzy A on a non-empty G is an (4, 2)-fuzzy set of form

$A = \{ \langle x, \delta_A^4(x), 1 - \lambda_A^2(x) \rangle / x \in G \}$. For the sake of simplicity, we just write

$$A = \langle \delta_A^4, \lambda_A^4 \rangle \text{ Instead of } A = \{ \langle x, \delta_A(x), \lambda_A(x) \rangle / x \in G \}.$$

The (4,2)-fuzzy sets $0 \sim$ and $1 \sim$ in G are defined by $0 \sim = \{ \langle x, 0, 1 \rangle : x \in G \}$ and $1 \sim = \{ \langle x, 1, 0 \rangle : x \in G \}$, respectively.

If φ is a mapping which maps a set G_1 into another set G_2 , then the following statement hold:

- (a) If $B = \{ \langle y, \delta_B^4(y), \lambda_B^2(y) \rangle / y \in G_2 \}$ is an (4,2)-fuzzy set in G_2 , then the pre image of B under φ , denoted by $\varphi^{-1}(B)$, is still an (4,2)-fuzzy set in G_1 , we now write $\varphi^{-1}(B) = \{ \langle x, \varphi^{-1}(\delta_B)(x), \varphi^{-1}(\lambda_B)(x) \rangle / x \in G_1 \}$.
- (b) If $A = \{ \langle x, \delta_A^4(x), \lambda_A^2(x) \rangle / x \in G_1 \}$ in an (4,2)-fuzzy set in G_1 , then the image of A under φ , denoted by $\varphi(A)$, is also an (4,2)fuzzy set in G_2 , which is defined by

$$\varphi(A) = \{ \langle y, \varphi_{\text{sup}}(\delta_A)(y), \varphi_{\text{inf}}(\lambda_A)(y) \rangle : y \in G_2 \},$$

where

$$\varphi_{\text{sup}}(\delta_A)(y) = \begin{cases} \sup_{x \in \varphi^{-1}(y)} \delta_A(x), & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0, & \text{else where,} \end{cases}$$

$$\varphi_{\text{inf}}(\lambda_A)(y) = \begin{cases} \inf_{x \in \varphi^{-1}(y)} \lambda_A(x), & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0, & \text{else where,} \end{cases}$$

for each $y \in G_2$.

1.1.1 Proposition

Let $A, A_i (i \in I)$ be (4,2)-fuzzy set in G_1 and B an (4,2)-fuzzy set in G_2 . If $\varphi: G_1 \rightarrow G_2$ is a function, then the following properties hold for the function φ :

- (a) If φ is surjective, then $\varphi(\varphi^{-1}(B)) = B$.
- (b) $\varphi^{-1}(\cup_{i=1}^n A_i) = \cup_{i=1}^n \varphi^{-1}(A_i)$.
- (c) $\varphi^{-1}(1 \sim) = 1 \sim$.
- (d) $\varphi^{-1}(0 \sim) = 0 \sim$.
- (e) $\varphi(1 \sim) = 1 \sim$, if φ is surjective
- (f) $\varphi(0 \sim) = 0 \sim$.

1.1.2 Definition

An (4,2)-fuzzy topology on a non-empty set G is a family τ of (4,2)-fuzzy sets in G which satisfies the following conditions:

- (i) $0 \sim, 1 \sim \in \tau$.
- (ii) If $G_1, G_2 \in \tau$, then $G_1 \cap G_2 \in \tau$.
- (iii) If $G_j \in \tau$ for all $j \in J$, then $\cup_{i \in I} G_i \in \tau$.

The pair (G, τ) is called an (4,2)-fuzzy topological space and any (4,2)-fuzzy set in τ is called an (4,2)-fuzzy open sets in G . The topology τ on a (4,2)-fuzzy topological space is said to be an indiscrete (4,2)-fuzzy topology if it's only element are $0 \sim$ and $1 \sim$.

On the other hand, (4,2)-fuzzy topology τ on a space G is said to be discrete (4,2)-fuzzy topology if the topology (4,2)-fuzzy topology τ contains all (4, 2)-fuzzy subsets of G . If A is an (4,2)-fuzzy set in an (4,2)-fuzzy topological space (G, τ) , then the induced (4,2)-fuzzy topological space on A is the family of (4,2)-fuzzy sets in A which are the intersection with A of (4,2)-fuzzy sets in G . The induced (4,2)-fuzzy topology is denoted by τ_A , and the pair (A, τ_A) is called an fuzzy subspace of (G, τ) . Let (G_1, τ_1) and (G_2, τ_2) be two (4,2)-fuzzy topological spaces and $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ a function. Then φ is said to be (4,2)-fuzzy continuous function if and only if the pre image of each (4,2)-fuzzy set in τ_2 is an (4,2)-fuzzy set in τ_1 . Let (G_1, τ_1) and (G_2, τ_2) be two (4,2)-fuzzy topological spaces and $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ a function. Then φ is said to be (4,2)-fuzzy open if and only if the image of each (4,2)-fuzzy set in τ_1 is an (4,2)-fuzzy set in τ_2 .

2. (4,2)-fuzzy topological sub algebras

2.1 Definition

An (4,2)-fuzzy set $A = \langle \delta_A^4, \lambda_A^2 \rangle$ in G is called (4,2)-fuzzy sub algebra of G if it satisfies the following conditions;

$$\text{FS1} : \delta_A^4(x * y) \geq \min\{\delta_A^4(x), \delta_A^4(y)\}$$

$$\text{FS2} : \lambda_A^2(x * y) \leq \max\{\lambda_A^2(x), \lambda_A^2(y)\}, \text{ for all } x, y \in G.$$

2.2 Example

Let $G = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the following Cayley table: 1.

Table: 1 BCC-algebra

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $A = \langle \delta_A^4, \lambda_A^2 \rangle$ be an (4,2)-fuzzy set in G defined by $\delta_A^4(4) = 0.07, \delta_A^4(x) = 0.6,$
 $\lambda_A^2(x) = 0.5$ and $\lambda_A^2(4) = 0.06$ for all $x \neq d$. Then A is (4,2)-fuzzy sub algebra of G .

2.3 Definition

Let τ_1 and τ_2 be an (4,2)-fuzzy topologies on BCC-algebras G_1 and G_2 respectively. A function $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is called an (4,2)-fuzzy continuous function which maps (G_1, τ_1) and (G_2, τ_2) if φ satisfies the following conditions:

- (i) For every $A \in \tau_2, \varphi^{-1}(A) \in \tau_1$.
- (ii) For every (4,2)-fuzzy sub algebra A (of G_2) in $\tau_2, \varphi^{-1}(A)$ is (4,2)-fuzzy sub algebra (of G_1) in τ_1 .

2.4 Proposition

If in τ_1 is an (4,2)-fuzzy topology on a BCC-algebra G_1 and τ_2 is an (4,2)-fuzzy topology on a BCC-algebra G_2 , then every function $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is a (4,2)-fuzzy continuous function. Proof: Since τ_2 is an indiscrete (4,2)-fuzzy topology, $\tau_2 = \{0\sim, 1\sim\}$. Let $\varphi: G_1 \rightarrow G_2$ be any mapping of BCC-algebras. Then, every member of τ_2 is an (4,2)-fuzzy topology on a BCC-algebra G_2 . We now show that φ is (4,2)-fuzzy continuous function. We only need to prove that for every $A \in \tau_2, \varphi^{-1}(A) \in \tau_1$. For this purpose, we let $0\sim \in \tau_2$. Then for any $x \in G_1$, we have $\varphi^{-1}(0\sim)(x) = 0\sim(\varphi(x)) = 0 = 0\sim(x)$
 This show that $(\varphi^{-1}(0\sim)) = 0\sim \in \tau_1$.

On the other hand, if $1\sim \in \tau_2$ and $x \in G_1$, then $\varphi^{-1}(1\sim)(x) = 1\sim(\varphi(x)) = 1 = 1\sim(x)$.

Thus $(\varphi^{-1}(1\sim)) = 1\sim \in \tau_1$.

This show that φ is indeed an (4,2)-fuzzy continuous function of G_1 to G_2 .

2.5 Theorem

Let τ_1 and τ_2 be any two discrete (4,2)-fuzzy topologies defined on the BCC-algebras G_1 and G_2 respectively. Then every homomorphism $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is an (4,2)-fuzzy continuous function.

Proof: Since τ_1 and τ_2 are discrete (4,2)-fuzzy topologies on the BCC-algebras G_1 and G_2 respectively, we have $\varphi^{-1}(A) \in \tau_1$ for every $A \in \tau_2$. We note that φ is not the usual inverse homomorphism from G_2 to G_1 .

Let $A = \langle \delta_A^4, \lambda_A^2 \rangle$ be an (4,2)-fuzzy sub algebra (of G_2) in τ_2 . Then for $x, y \in G_1$, we have, $(\varphi^{-1}(\delta_A^4))(x * y) = \delta_A^4(\varphi(x * y))$
 $= \delta_A^4(\varphi(x) * \varphi(y))$
 $\geq \min\{\delta_A^4(\varphi(x)), \delta_A^4(\varphi(y))\}$
 $= \min\{(\varphi^{-1}(\delta_A^4))(x), (\varphi^{-1}(\delta_A^4))(y)\}$ and
 $(\varphi^{-1}(\lambda_A^2))(x * y) = \lambda_A^2(\varphi(x * y))$
 $= \lambda_A^2(\varphi(x) * \varphi(y))$
 $\leq \max\{\lambda_A^2(\varphi(x)), \lambda_A^2(\varphi(y))\}$
 $= \max\{(\varphi^{-1}(\lambda_A^2))(x), (\varphi^{-1}(\lambda_A^2))(y)\}$

Hence $\varphi^{-1}(A)$ is an (4,2)-fuzzy sub algebra (of G_1) in τ_1 and consequently, φ is an (4,2)-fuzzy continuous function which maps (G_1, τ_1) to (G_2, τ_2) .

2.6 Definition:

Let (G_1, τ_1) and (G_2, τ_2) be (4,2)-fuzzy topology sub algebras. A function $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ is said to be an (4,2)-fuzzy homomorphism if it satisfies the following conditions:

- φ is an injective and surjective function.
- φ is fuzzy continues function which maps G_1 to G_2 .
- φ^{-1} is fuzzy continues function which maps G_2 to G_1 .

2.7 Definition

Let τ be an (4,2)-fuzzy topology of BCC-algebra G . An (4,2)-fuzzy topology (G, τ) is an (4,2)- fuzzy Hausdorff space if and only if for any discrete (4,2)-fuzzy point $x_1, x_2 \in G$, there exists (m, n) - fuzzy topology $F_1 = \langle \delta_{F_1}^4, \lambda_{F_1}^2 \rangle$ and $F_2 = \langle \delta_{F_2}^4, \lambda_{F_2}^2 \rangle$ such that $\delta_{F_1}^4(x_1) = 1$, $\lambda_{F_1}^2(x_1) = 0$, $\delta_{F_2}^4(x_2) = 1$, $\lambda_{F_2}^2(x_2) = 0$ and $F_1 \cap F_2 = 0 \sim$.

2.8 Theorem

Let τ_1 and τ_2 be (4,2)-fuzzy topologies on the BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be an (4,2)-fuzzy homomorphism. Then G_1 is an (4,2)-fuzzy Hausdorff space if and only if G_2 is an (4,2)-fuzzy Hausdorff space. Proof: Suppose that G_1 is a (4,2)-fuzzy Hausdorff space.

Let x_1, x_2 be the (4,2)-fuzzy point in τ_2 with $x \neq y$ where $x, y \in G_1$.

Then $\varphi^{-1}(x) \neq \varphi^{-1}(y)$ because φ is injective function.

For $z \in G_1$, $(\varphi^{-1}(x_1))(z) = x_1 (\varphi(z))$

$$= \begin{cases} s \in [0, 1], & \text{if } \varphi(z) = x \\ 0, & \text{if } \varphi(z) \neq x \end{cases} = \begin{cases} s \in [0, 1], & \text{if } z = \varphi^{-1}(x) \\ 0, & \text{if } z \neq \varphi^{-1}(x) \end{cases} = (\varphi^{-1}(x))_1(z).$$

That is, $(\varphi^{-1}(x_1))(z) = (\varphi^{-1}(x))_1(z)$ for all $z \in G$. Hence $\varphi^{-1}(x_1) = (\varphi^{-1}(x))_1$. Similarly we can also prove that $\varphi^{-1}(x_2) = (\varphi^{-1}(x))_2$. Now by the definition of an (4,2)-fuzzy Hausdorff space, there exist (4,2)-fuzzy order F_1 and F_2 of $\varphi^{-1}(x_1)$ and $\varphi^{-1}(x_2)$ respectively such that $F_1 \cap F_2 = 0 \sim$. Since φ is an (4,2)-fuzzy continuous map from G_2 to G_1 , there exist (4,2)-fuzzy orders $\varphi(F_1)$ and $\varphi(F_2)$ of x_1 and x_2 respectively such that $\varphi(F_1) \cap \varphi(F_2) = \varphi(F_1 \cap F_2) = \varphi(0 \sim) = 0 \sim$. This implies that G_2 is a (4,2)-fuzzy Hausdorff space. Conversely, if (G_2, τ_2) is a (4,2)-fuzzy Hausdorff space, then by using a similar argument as above and by the fact that both φ and φ^{-1} are (4,2)-fuzzy continuous functions, we can easily prove that (G_1, τ_1) is an (4,2)-fuzzy Hausdorff space. Hence the proof.

2.9 Definition

Let τ be an (m, n) - fuzzy topology on a BCC-algebra G . Then (G, τ) is called an (4,2)-fuzzy C_5 -disconnected space if there exists an (4,2)-fuzzy open and closed set F such that $F \neq 0 \sim$ and $F \neq 1 \sim$.

2.10 Theorem

Let τ_1 and τ_2 be the (4,2)-fuzzy topology sub algebras G_1 and G_2 respectively and let $\varphi: G_1 \rightarrow G_2$ be an (4,2)-fuzzy continuous surjective function. If (G_1, τ_1) is an (4,2)-fuzzy C_5 -connected space then (G_2, τ_2) is also an (4,2)-fuzzy C_5 -connected space.

Proof: Assume that (G_2, τ_2) is a (4,2)- fuzzy C_5 -disconnected. Then there exist an (4,2)-fuzzy open and closed set F such that $F \neq 0 \sim$ and $F \neq 1 \sim$. Since φ is an (m, n) - fuzzy continuous function $\varphi^{-1}(F)$ is

both (4,2)-fuzzy open and (4,2)-fuzzy closed set. In this case $\varphi^{-1}(F) \neq 0\sim$ or $\varphi^{-1}(F) \neq 1\sim$.

Since, $F = \varphi(\varphi^{-1}(F)) = \varphi(0\sim) = 0\sim$ and $F = \varphi(\varphi^{-1}(F)) = \varphi(1\sim) = 1\sim$. We see that these results contradict to our assumption. Hence the space (G_2, τ_2) must be (4,2)-fuzzy C_5 -connected space.

2.11 Definition

Let τ be an (4,2)-fuzzy topology on a BCC-algebra G . An (4,2)-fuzzy topology (G, τ) is called an (4,2)-fuzzy disconnected space if there exist (4,2)-fuzzy open sets $A \neq 0\sim$ and $B \neq 0\sim$ such that $A \cup B = 0\sim$. Naturally, we call the set (G, τ) an (m, n)-fuzzy connected if (G, τ) is not (4,2)-fuzzy disconnected.

2.12 Theorem

Let τ_1 and τ_2 be (m, n)-fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be an (4,2)-fuzzy continuous and surjective function. If G_1 is an (4,2)-fuzzy connected space, then so is G_2 . Proof: Suppose that G_2 is an (m, n)-fuzzy disconnected, then there exists (4,2)-fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ in G_2 such that $C \cup D = 1\sim$ and $C \cap D = 0\sim$. Since φ is (4,2)-fuzzy continuous function, $A = \varphi^{-1}(C)$ and $B = \varphi^{-1}(D)$ are (m, n)-fuzzy open sets in G_1 . Clearly, $C \neq 0\sim$ implies that $A = \varphi^{-1}(C) \neq 0\sim$, and $D \neq 0\sim$ implies that $B = \varphi^{-1}(D) \neq 0\sim$. Now $C \cup D = 1\sim$.

$$\begin{aligned} &\Rightarrow \varphi^{-1}(C \cup D) = \varphi^{-1}(1\sim). \\ &\Rightarrow \varphi^{-1}(C) \cup \varphi^{-1}(D) = 1\sim \text{ implies } A \cup B = 1\sim \\ &\text{and } C \cap D = 0\sim \Rightarrow \varphi^{-1}(C \cap D) = \varphi^{-1}(0\sim) \\ &\Rightarrow \varphi^{-1}(C) \cap \varphi^{-1}(D) = 0\sim \text{ implies } A \cap B = 0\sim. \end{aligned}$$

This clearly contradicts our hypothesis.

Hence G_2 is an (4,2)-fuzzy connected space.

2.13 Definition

An (m, n)-fuzzy topology space (G, τ) is said to be an (4,2)-fuzzy strongly connected, if there exists no non-zero (4,2)-fuzzy closed sets A and B in G such that $\delta_A^4 + \delta_B^4 \leq 1$ and $\lambda_A^2 + \lambda_B^2 \geq 1$.

The following fact follows immediately from the definition.

2.14 Proposition

G is (4,2)-fuzzy strongly connected if and only if there exist an (4,2)-fuzzy open sets A and B in G such that $A \neq 1\sim \neq B$ and $\delta_A^4 + \delta_B^4 \geq 1$, $\lambda_A^2 + \lambda_B^2 \leq 1$. We now formulate the following theorem.

2.15 Theorem

Let τ_1 and τ_2 be (4,2)-fuzzy topology set on BCC-algebras G_1 and G_2 respectively and let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be an (4,2)-fuzzy continuous and surjective mapping. If G_1 is an (4,2)-fuzzy strongly connected, then so is G_2 .

Proof: Suppose that G_2 is not an (4,2)-fuzzy strongly connected. Then there exists (4,2)-fuzzy open sets $C \neq 0\sim$ and $D \neq 0\sim$ so that $\delta_C^4 + \delta_D^4 \leq 1$ and $\lambda_C^2 + \lambda_D^2 \geq 1$. Since φ is an (4,2)-fuzzy continuous function, $\varphi^{-1}(C)$ and $\varphi^{-1}(D)$ are (4,2)-fuzzy closed sets in G_1 . Now we can deduce the following equalities;

$$\begin{aligned} \delta_{\varphi^{-1}(C)}^4 + \delta_{\varphi^{-1}(D)}^4 &= \varphi^{-1}(\delta_C^4) + \varphi^{-1}(\delta_D^4) \\ &= \delta_C^4 \circ \varphi + \delta_D^4 \circ \varphi \leq 1 \text{ (Since } \delta_C^4 + \delta_D^4 \leq 1), \\ \lambda_{\varphi^{-1}(C)}^2 + \lambda_{\varphi^{-1}(D)}^2 &= \varphi^{-1}(\lambda_C^2) + \varphi^{-1}(\lambda_D^2) \\ &= \lambda_C^2 \circ \varphi + \lambda_D^2 \circ \varphi \geq 1 \text{ (Since } \lambda_C^2 + \lambda_D^2 \geq 1). \end{aligned}$$

$\varphi^{-1}(C) \neq 0\sim$ and $\varphi^{-1}(D) \neq 0\sim$. This contradicts our hypothesis. Hence G_2 is an (4,2)-fuzzy strongly connected space.

2.16 Definition

Let τ be an (4,2)-fuzzy topology on a BCC-algebra G and A be an (4,2)-fuzzy BCC-algebra with (4,2)-

fuzzy topology τ_A . Then A is called an (4,2)-fuzzy topological BCC-sub algebra if the self-mapping $\gamma_a: (A, \tau_A) \rightarrow (A, \tau_A)$ defined by $\gamma_a(x) = x * a$ for all $x \in G$, is a Relatively (4,2)-fuzzy continuous function.

2.17 Theorem

Let $\varphi: G_1 \rightarrow G_2$ be a homomorphism of BCC-algebras and τ and τ^* be (4,2)-fuzzy topologies on G_1 and G_2 respectively such that $\tau = \varphi^{-1}(\tau^*)$. If B is an (4,2)-fuzzy topological BCC-sub algebra in G_2 , then $\varphi^{-1}(B)$ is an (4,2)-fuzzy topological BCC-sub algebra in G_1 .

2.18 Theorem

Let $\varphi: G_1 \rightarrow G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively (4,2)-fuzzy topologies on the spaces G_1 and G_2 such that $\tau = \varphi^{-1}(\tau^*)$. If A is an (4,2)-fuzzy topological BCC-sub algebra in G_1 , then $\varphi^{-1}(A)$ is an (4,2)-fuzzy topological BCC-sub algebra in G_2 .

3. (4,2)-fuzzy topological BCC-ideals

3.1 Definition

An (4,2)-fuzzy set $A = \{\{\delta_A, \lambda_A\}\}$ in a BCK-algebra G is called an (4,2)-fuzzy BCK-ideal of G if the following conditions are satisfied;

- (i) $\delta_A^4(0) \geq \delta_A^4(x)$ and $\lambda_A^2(0) \leq \lambda_A^2(x)$,
- (ii) $\delta_A^4(x) \geq \min\{\delta_A^4(x * y), \delta_A^4(y)\}$
- (iii) $\lambda_A^2(x) \leq \max\{\lambda_A^2(x * y), \lambda_A^2(y)\}$ for all $x, y \in G$.

3.2 Definition

An (4,2)-fuzzy set $A = \langle \delta_A, \lambda_A \rangle$ in G is called an (4,2)-fuzzy BCC-ideal of G if it satisfies the following conditions;

- (4,2) F₁: $\delta_A^4(0) \geq \delta_A^4(x)$ and $\lambda_A^2(0) \leq \lambda_A^2(x)$
 - (4,2) F₂: $\delta_A^4(x * z) \geq \min\{\delta_A^4((x * y) * z), \delta_A^4(y)\}$
 - (4,2) F₃: $\lambda_A^2(x * z) \leq \max\{\lambda_A^2((x * y) * z), \lambda_A^2(y)\}$
- for all $x, y, z \in G$.

Putting $z = 0$ in (m, n) F₂ and (4,2) F₃, then we can easily see that an (4,2)-fuzzy BCC-ideal is an (m, n)-fuzzy BCK-ideal. However, the converse does not hold.

3.3 Example

Let $G = \{0, 1, 2, 3, 4, 5\}$ be a BCC-algebra with the following Cayley table: 2.

Table: 2 BCC-Algebra

+	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Let $A = \langle \delta_A, \lambda_A \rangle$ be an (4,2)-fuzzy set in G defined by $\delta_A^4(5) = 0.02$, $\delta_A^4(x) = 0.4$, $\lambda_A^2(5) = 0.2$ and $\lambda_A^2(x) = 0.04$ for all $x \neq 5$, then A is an (4,2)-fuzzy BCC-ideal of a BCC-algebra G.

3.4 Theorem: Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an (4,2)-fuzzy BCC-ideal of G_2 . Then $\varphi^{-1}(B)$ is an (4,2)-fuzzy BCC-ideal of G_1 .

Proof: It can be easily seen that $\delta_{\varphi^{-1}(B)}^4(0) \geq \delta_{\varphi^{-1}(B)}^4(x)$ and $\lambda_{\varphi^{-1}(B)}^2(0) \leq \lambda_{\varphi^{-1}(B)}^2(x)$, for all $x \in G_1$.

For any $x, y, z \in G_1$, we can deduce the following

$$\begin{aligned} \delta_{\varphi^{-1}(B)}^4(x * z) &= \delta_B^4(\varphi(x * z)) \\ &\geq \min \left\{ \delta_B^4(\varphi((x * y) * z)), \delta_B^4(\varphi(y)) \right\} \\ &= \min \left\{ \delta_B^4((\varphi(x) * \varphi(y)) * \varphi(z)), \delta_B^4(\varphi(y)) \right\} \\ &= \min \left\{ \delta_{\varphi^{-1}(B)}^4((x * y) * z), \delta_{\varphi^{-1}(B)}^4(y) \right\}. \text{ Also} \\ \lambda_{\varphi^{-1}(B)}^2(x * z) &= \lambda_B^2(\varphi(x * z)) \\ &\leq \max \left\{ \lambda_B^2(\varphi((x * y) * z)), \lambda_B^2(\varphi(y)) \right\} \\ &= \max \left\{ \lambda_B^2((\varphi(x) * \varphi(y)) * \varphi(z)), \lambda_B^2(\varphi(y)) \right\} \\ &= \max \left\{ \lambda_{\varphi^{-1}(B)}^2((x * y) * z), \lambda_{\varphi^{-1}(B)}^2(y) \right\} \end{aligned}$$

Hence $\varphi^{-1}(B)$ is an (4,2)-fuzzy BCC-ideal of G_1 .

3.5 Corollary

Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 and B be an (4,2)-fuzzy BCK-ideal of G_2 . Then $\varphi^{-1}(B)$ is an (4,2)-fuzzy BCK-ideal of G_1 . Since an (m, n)-fuzzy BCC-ideal / BCK-ideal is an (4,2)-fuzzy sub algebra, as a consequence of the above results and theorem-3.17, we obtain the following corollary.

3.6 Corollary

Let $\varphi: (G_1, \tau_1) \rightarrow (G_2, \tau_2)$ be a homomorphism of the BCC-algebras. Let τ_1 and τ_2 be the (4,2)-fuzzy topologies on G_1 and G_2 respectively such that $\tau_2 = \varphi^{-1}(\tau_1)$. If B is (4,2)-fuzzy topological BCC-ideal / BCK-ideal of G_2 with the membership function δ_B^4 , then $\varphi^{-1}(B)$ is a (4,2)-fuzzy topological BCC-ideal / BCK-ideal of G_1 with the membership function $\delta_{\varphi^{-1}(B)}^m$.

3.7 Theorem

Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 . If A is an (4,2)-fuzzy BCC-ideal

of G_1 , then the homomorphic image $\varphi(A)$ of A is still an (4,2)-fuzzy BCC-ideal of G_2 .

Proof: Let A be an (4,2)-fuzzy topological BCC-ideal of G_1 . Then, it is trivial that

$\delta_{\varphi(A)}^4(0) \geq \delta_{\varphi(A)}^4(x)$ and $\lambda_{\varphi(A)}^2(0) \leq \lambda_{\varphi(A)}^2(x)$, for all $x \in G_2$.

Take $x, y, z \in G_2$, and let $x_0 \in \varphi^{-1}(x), y_0 \in \varphi^{-1}(y), z_0 \in \varphi^{-1}(z)$ such that

$$\begin{aligned} \delta_A^4(x_0) &= \sup_{t \in \varphi^{-1}(x)} t, & \delta_A^4(y_0) &= \sup_{t \in \varphi^{-1}(y)} t \text{ and} \\ \delta_A^4(z_0) &= \sup_{t \in \varphi^{-1}(z)} t. \end{aligned}$$

Then we can deduce the following,

$$\begin{aligned} \delta_{\varphi(A)}^4(x * z) &= \sup_{t \in \varphi^{-1}(x * z)} (\delta_A^4(t)) \geq \delta_A^4(x_0 * z_0) \\ &\geq \min \{ \delta_A^4((x_0 * y_0) * z_0), \delta_A^4(y_0) \} = \\ &\min \left\{ \sup_{t \in \varphi^{-1}((x * y) * z)} (\delta_A^4(t)), \sup_{t \in \varphi^{-1}(y)} (\delta_A^4(t)) \right\} \\ &= \min \{ \delta_{\varphi(A)}^4((x * y) * z), \delta_{\varphi(A)}^4(y) \} \\ \text{and } \lambda_{\varphi(A)}^2(x * z) &= \inf_{t \in \varphi^{-1}(x * z)} (\lambda_{\varphi(A)}^2(t)) \leq \\ &\lambda_A^2(x_0 * z_0) \\ &\leq \max \{ \lambda_A^2((x_0 * y_0) * z_0), \lambda_A^2(y_0) \} \\ &= \max \left\{ \inf_{t \in \varphi^{-1}((x * y) * z)} (\lambda_A^2(t)), \inf_{t \in \varphi^{-1}(y)} (\lambda_A^2(t)) \right\} \\ &= \max \{ \lambda_{\varphi(A)}^2((x * y) * z), \lambda_{\varphi(A)}^2(y) \} \end{aligned}$$

Hence $\varphi(A) = \langle \varphi_{\sup}(\delta_A), \varphi_{\inf}(\lambda_A) \rangle$ is induced an (4,2)-fuzzy BCC-ideal of G_2 .

Putting $z = 0$ in the above theorem, we obtain:

3.8 Corollary

Let φ be a homomorphism of a BCC-algebra G_1 into a BCC-algebra G_2 . If A is an (4,2)-fuzzy BCK-ideal of G_1 , then the homomorphic image $\varphi(A)$ of A is still an (4,2)-fuzzy BCK-ideal of G_2 . Summing up theorem-3.18, theorem-4.7 and corollary-4.8, we conclude the following theorem.

3.9 Theorem

Let $\varphi: G_1 \rightarrow G_2$ be an isomorphism of BCC-algebras. Let τ and τ^* be the respectively (4,2)-fuzzy topologies on the spaces G_1 and G_2 such that $\varphi(\tau) = \tau^*$. If A is an (4,2)-fuzzy topological BCC-ideal / BCK-ideal in G_1 , then $\varphi(A)$ is also an (4,2)-fuzzy topological BCC-ideal / BCK-ideal in G_2 .

4. Conclusion

We study the concept of (4,2)-fuzzy topological properties of such algebras such as connectedness, strong connectedness and compact Hausdorff space. We also discussed the characteristic of the homomorphic image and inverse image of (4,2)-fuzzy topological BCC-ideals (BCK-ideals) of BCC-algebras (BCK-algebras).

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