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ADAMS EXPLICIT APPROACH FOR THE NUMERICAL SOLUTION OF SINGLE-ORDERED FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract

Fractional Differential Equations have lately enthralled many researchers due to its emerging use in almost every science related field. The aim of this paper is to derive fractional four step explicit Adams Bashforth method for solving some linear and nonlinear fractional differential equations of order α∈(0,1) where the fractional derivatives are defined in Caputo sense. The numerical results for the derived method are compared with the exact solution for linear and nonlinear fractional differential equations by using absolute error at each integration point. The behavior of calculated solutions is tabulated and plotted at different values of fractional order α. Matlab is used in completing the required steps of the above procedures.

1. Introduction

Differential equations in calculus have earned great significance due to their ability to model the real-world phenomena. However, these differential equations are not only confined to integer order but also extended to arbitrary order that we study in fractional calculus. In recent decades, Fractional Calculus and Fractional Differential Equations (FDEs) have led many mathematicians to propose both analytical and numerical methods for solving these equations due to their emerging applications in many scientific and engineering areas. Several numerical schemes have been proposed for solving FDEs (Diethelm *et al.*, 2002, Diethelm, 2010, Garrappa, 2009, Galeone *et al.*, 2006, Jator *et al.*, 2015, Li *et al.*, 2013 & Li *et al.*, 2009). For both linear and nonlinear FDEs, the approach has been shown to have the advantage of being simple to implement. However, the function evaluation for the iterations will rise as the number of intervals N increases. In (Kilbas *et al.,* 2006), an explicit technique for solving fractional order partial differential equations based on the Adams–Bashforth numerical scheme in a Laplace space is proposed. As most of FDEs do not have analytic solutions, we have to use methods to convert them to more accurate equations, like Volterra integral equations. For which we can use various approximations and numerical techniques (Diethelm *et al.,* 2002, Diethelm *et al.,* 2004 & Diethelm, 2010). For review of issues of fractional calculus, we refer the readers to classical books on the subject (Kilbas *et al.,* 2006, Miller *et al.,* 1993 & Podlubny, 1999). Despite analytical methods, many researchers have concluded that numerical methods are more convenient and efficient way to solve FDEs, particularly in cases where analytical solutions are not possible. Consequently, a significant amount of research has been published on the numerical solutions of FDEs.

2. Materials and Methods

In order, to propose fractional four step explicit Adams Bashforth method (FFSEABM) we shall use explicit Adams Bashforth method. We focus our attention to the fractional initial value problem (FIVP)

$$D_C^{\alpha} y(t) = f\big(t, y(t)\big), \qquad y^k(t_0) = {y_0}^k, k = 0,1, \dots, \lceil \alpha \rceil - 1. \tag{1}$$

Where $\alpha > 0$, is the order of FDE, ${y_0}^k$ may be any real number , D_C^{α} denotes the Caputo's αth order derivative. Then Lagrange Interpolation with four interpolating functions, that are F_n, F_{n-1}, F_{n-2} *and* F_{n-3}, is implemented on the equation. Integration techniques are used to obtain the required explicit formula.

3. Results

In this section, we derive the FFSEABM. It would be useful to introduce some definitions and properties of fractional calculus.

Definition 3.1: [3]

The αth-order Caputo fractional derivative of $y(t)$ is defined as:

$$D_C^{\alpha} y(p) = \frac{1}{\Gamma(m-\alpha)} \int_{p_0}^{p} \frac{y^m(\tau) d\tau}{(p-\tau)^{\alpha-m+1}}, \qquad m-1 < \alpha < m \in Z^+.$$

where $n = \lceil \alpha \rceil$ is the first integer not less than α and y^n denotes the nth integer order derivative of y.

Lemma 3.1 [3]:

If the function f is continuous, then the fractional initial value problem is equivalent to the Volterra integral equation,

$$y(t) = \sum_{m=0}^{\lceil \alpha \rceil - 1} \frac{t^m}{m!} {y_0}^m + \frac{1}{\Gamma(\alpha)} \int_0^t \left[(t-\lambda)^{\alpha-1} f\big(\lambda, y(\lambda)\big)\right] d\lambda. \qquad \dots (1)$$

With $n = \lceil \alpha \rceil$ and $n - 1 < \alpha < n.$

Construction of FFSEABM:

We start with FIVP in the following form:

$$D_C^{\alpha} y(t) = f\big(t, y(t)\big), \qquad y^k(t_0) = {y_0}^k, k = 0,1, \dots, \lceil\alpha\rceil - 1. \qquad \dots (2)$$

Where $\alpha > 0$, is the order of FDE, ${y_0}^k$ may be any real number , D_C^{α} denotes the Caputo's αth order derivative and $f: [0,T] \times R \to R$ for $t \in [t_0, T]$. In order, to assure the existence and uniqueness of the solution to equation (2), it is assumed that $f\big(t, y(t)\big)$ is continuous and fulfills the Lipschitz condition with respect to the second variables. On [0,T], for a uniform grid $t_j = hj$, $(j = 0,1, \dots, N)$ and a constant step size denoted by $h = \frac{T}{N}$, the goal is to approximate solutions $y_j \approx y(t_j)$ at the grid points. It is well known that the IVP (2) is equivalent to Volterra Integral equation [Lemma 3.1].

$$y(t) = \sum_{m=0}^{\lceil\alpha\rceil-1} \frac{t^m}{m!} {y_0}^m + \frac{1}{\Gamma(\alpha)} \int_0^t \big[(t-\lambda)^{\alpha-1} f\big(\lambda, y(\lambda)\big)\big]\, d\lambda. \qquad \dots (3)$$

In the sense that a continuous function y(t) is a solution of (2) if and only if it is a solution of equation (3). For 0 < α < 1, equation (3) reduces to

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \big[(t-\lambda)^{\alpha-1} f\big(\lambda, y(\tau)\big)\big]\, d\tau. \qquad \dots (4)$$

For $t = t_n$, from (4) we have

$$y(t_n) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \big[(t_n-\lambda)^{\alpha-1} f\big(\lambda, y(\tau)\big)\big]\, d\tau. \qquad \dots (5)$$

For $t = t_{n+1}$, from equation (4) we have

$$y(t_{n+1}) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \big[(t_{n+1}-\lambda)^{\alpha-1} f\big(\lambda, y(\tau)\big)\big]\, d\tau. \qquad \dots (6)$$

Subtracting equations (5) and (6) we get:

$$y(t_{n+1}) = y(t_n) + \frac{1}{\Gamma(\alpha)} \Big[\int_0^{t_{n+1}} \big[(t_{n+1}-\lambda)^{\alpha-1} f\big(\lambda, y(\lambda)\big)\big]\, d\lambda - \int_0^{t_n} \big[(t_n-\lambda)^{\alpha-1} f\big(\lambda, y(\lambda)\big)\big]\, d\lambda\Big]. \qquad \dots (7)$$

In order to propose the fractional four step explicit Adams Bashforth method (FFSEABM) we interpolate $f(\lambda, y(\lambda))$ using Lagrange Interpolation with four interpolating functions F_n, F_{n-1}, F_{n-2} and F_{n-3} as follow:

$$P(t) \approx f\big(\lambda, y(\lambda)\big) = \frac{(t-t_{n-1})(t-t_{n-2})(t-t_{n-3})}{(t_n-t_{n-1})(t_n-t_{n-2})(t_n-t_{n-3})} F_n + \frac{(t-t_n)(t-t_{n-2})(t-t_{n-3})}{(t_{n-1}-t_n)(t_{n-1}-t_{n-2})(t_{n-1}-t_{n-3})} F_{n-1} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-3})}{(t_{n-2}-t_n)(t_{n-2}-t_{n-1})(t_{n-2}-t_{n-3})} F_{n-2} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-2})}{(t_{n-3}-t_n)(t_{n-3}-t_{n-1})(t_{n-3}-t_{n-2})} F_{n-3}. \qquad \dots (8)$$

Also let $h = t_{n+1} - t_n$, $\lambda = t$, … (9)

substituting equations (8) and (9) into equation (7), we obtain

$$y(t_{n+1}) = y(t_n) + \frac{1}{\Gamma(\alpha)}[\int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} (\frac{(t-t_{n-1})(t-t_{n-2})(t-t_{n-3})}{(t_n - t_{n-1})(t_n - t_{n-2})(t_n - t_{n-3})} F_n + \frac{(t-t_n)(t-t_{n-2})(t-t_{n-3})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})(t_{n-1} - t_{n-3})} F_{n-1} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-3})}{(t_{n-2} - t_n)(t_{n-2} - t_{n-1})(t_{n-2} - t_{n-3})} F_{n-2} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-2})}{(t_{n-3} - t_n)(t_{n-3} - t_{n-1})(t_{n-3} - t_{n-2})} F_{n-3})\, dt\,] - \frac{1}{\Gamma(\alpha)}[\int_0^{t_n} [\,(t_n - t)^{\alpha-1} (\frac{(t-t_{n-1})(t-t_{n-2})(t-t_{n-3},)}{(t_n - t_{n-1})(t_n - t_{n-2})(t_n - t_{n-3})} F_n + \frac{(t-t_n)(t-t_{n-2})(t-t_{n-3})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})(t_{n-1} - t_{n-3})} F_{n-1} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-3})}{(t_{n-2} - t_n)(t_{n-2} - t_{n-1})(t_{n-2} - t_{n-3})} F_{n-2} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-2})}{(t_{n-3} - t_n)(t_{n-3} - t_{n-1})(t_{n-3} - t_{n-2})} F_{n-3})\, dt\,]. \quad \text{... (10)}$$

The first integral in equation (10) is evaluated as:

$$\int_0^{t_{n+1}} (t_{n+1} - \lambda)^{\alpha-1} f(\lambda, y(\lambda))\, d\lambda$$

$$= \sum_{p=0}^{n} \int_{t_p}^{t_{p+1}} (t_{n+1} - t)^{\alpha-1} [\frac{(t-t_{n-1})(t-t_{n-2})(t-t_{n-3})}{(t_n - t_{n-1})(t_n - t_{n-2})(t_n - t_{n-3})} F_n + \frac{(t-t_n)(t-t_{n-2})(t-t_{n-3})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})(t_{n-1} - t_{n-3})} F_{n-1} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-3})}{(t_{n-2} - t_n)(t_{n-2} - t_{n-1})(t_{n-2} - t_{n-3})} F_{n-2} + \frac{(t-t_n)(t-t_{n-1})(t-t_{n-2})}{(t_{n-3} - t_n)(t_{n-3} - t_{n-1})(t_{n-3} - t_{n-2})} F_{n-3}]dt.$$

$$= \sum_{p=0}^{n} [\frac{F_n}{(t_n - t_{n-1})(t_n - t_{n-2})(t_n - t_{n-3})} \int_{t_p}^{t_{p+1}} (t_{n+1} - t)^{\alpha-1} (t-t_{n-1})(t-t_{n-2})(t-t_{n-3})dt$$

$$+ \frac{F_{n-1}}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})(t_{n-1} - t_{n-3})} \int_{t_p}^{t_{p+1}} (t_{n+1} - t)^{\alpha-1} (t-t_n)(t-t_{n-2})(t-t_{n-3})dt$$

$$+ \frac{F_{n-2}}{(t_{n-2} - t_n)(t_{n-2} - t_{n-1})(t_{n-2} - t_{n-3})} \int_{t_p}^{t_{p+1}} (t_{n+1} - t)^{\alpha-1} (t-t_n)(t-t_{n-1})(t-t_{n-3})dt$$

$$+ \frac{F_{n-3}}{(t_{n-3} - t_n)(t_{n-3} - t_{n-1})(t_{n-3} - t_{n-2})} \int_{t_p}^{t_{p+1}} (t_{n+1} - t)^{\alpha-1} (t-t_n)(t-t_{n-1})(t-t_{n-2})dt].$$

$$= \sum_{p=0}^{n} [\frac{F_n}{6h^3} \int_{t_p}^{t_{p+1}} (t_{n+1}-t)^{\alpha-1} (t-t_{n-1})(t-t_{n-2})(t-t_{n-3})dt$$

$$- \frac{F_{n-1}}{2h^3} \int_{t_p}^{t_{p+1}} (t_{n+1}-t)^{\alpha-1} (t-t_n)(t-t_{n-2})(t-t_{n-3})dt$$

$$+ \frac{F_{n-2}}{2h^3} \int_{t_p}^{t_{p+1}} (t_{n+1}-t)^{\alpha-1} (t-t_n)(t-t_{n-1})(t-t_{n-3})dt$$

$$- \frac{F_{n-3}}{6h^3} \int_{t_p}^{t_{p+1}} (t_{n+1}-t)^{\alpha-1} (t-t_n)(t-t_{n-1})(t-t_{n-2})dt].$$

Next, we implement the following change of variable:

$$y = t_{n+1} - t, \quad dy = -dt.$$

$$= \sum_{p=0}^{n} \{\frac{F_n}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_{n-1})(t_{n+1}-y-t_{n-2})(t_{n+1}-y-t_{n-3})(-dy)]$$

$$- \frac{F_{n-1}}{2h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_n)(t_{n+1}-y-t_{n-2})(t_{n+1}-y-t_{n-3})(-dy)]$$

$$+ \frac{F_{n-2}}{2h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_n)(t_{n+1}-y-t_{n-1})(t_{n+1}-y-t_{n-3})(-dy)]$$

$$- \frac{F_{n-3}}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_n)(t_{n+1}-y-t_{n-1})(t_{n+1}-y-t_{n-2})(-dy)]\}. \quad \ldots (11)$$

Let

$$I_1 = \sum_{p=0}^{n} \{\frac{F_n}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_{n-1})(t_{n+1}-y-t_{n-2})(t_{n+1}-y-t_{n-3})(-dy)]\}.$$

$$I_2 = \sum_{p=0}^{n} \{-\frac{F_{n-1}}{2h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_n)(t_{n+1}-y-t_{n-2})(t_{n+1}-y-t_{n-3})(-dy)]\}.$$

$$I_3 = \sum_{p=0}^{n} \{\frac{F_{n-2}}{2h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_n)(t_{n+1}-y-t_{n-1})(t_{n+1}-y-t_{n-3})(-dy)]\}.$$

$$I_4 = \sum_{p=0}^{n} \{-\frac{F_{n-3}}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (t_{n+1}-y-t_n)(t_{n+1}-y-t_{n-1})(t_{n+1}-y-t_{n-2})(-dy)]\}.$$

The computation for I_1 is given as:

$$= \sum_{p=0}^{n} \{-\frac{F_n}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (2h-y)(3h-y)(4h-y)dy]\}$$

$$= \sum_{p=0}^{n} \{-\frac{F_n}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} y^{\alpha-1} (24h^3 - 26h^2y + 9hy^2 - y^3)dy]\}$$

$$= \sum_{p=0}^{n} \{-\frac{F_n}{6h^3} [\int_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}} (24h^3y^{\alpha-1} - 26h^2y^{\alpha} + 9hy^{\alpha+1} - y^{\alpha+2})\, dy]\}$$

$$=\sum_{p=0}^{n}\{-\frac{F_n}{6h^3}[\frac{24h^3y^{\alpha}}{\alpha}-\frac{26h^2y^{\alpha+1}}{\alpha+1}+\frac{9hy^{\alpha+2}}{\alpha+2}-\frac{y^{\alpha+3}}{\alpha+3}]|_{t_{n+1}-t_p}^{t_{n+1}-t_{p+1}}\}$$

$$=\sum_{p=0}^{n}\{-\frac{F_n}{6h^3}[\frac{24h^3\left[t_{n+1}-t_{p+1}\right]^{\alpha}-\left[t_{n+1}-t_p\right]^{\alpha}}{\alpha}-\frac{26h^2\left[t_{n+1}-t_{p+1}\right]^{\alpha+1}-\left[t_{n+1}-t_p\right]^{\alpha+1}}{\alpha+1}$$
$$+\frac{9h\left[t_{n+1}-t_{p+1}\right]^{\alpha+2}-\left[t_{n+1}-t_p\right]^{\alpha+2}}{\alpha+2}-\frac{\left[t_{n+1}-t_{p+1}\right]^{\alpha+3}-\left[t_{n+1}-t_p\right]^{\alpha+3}}{\alpha+3}]\}$$

$$=\frac{F_n}{6h^3}[\frac{-24h^3[t_{n+1}-t_{n+1}]^{\alpha}-[t_{n+1}-t_0]^{\alpha}}{\alpha}+\frac{26h^2[t_{n+1}-t_{n+1}]^{\alpha+1}-[t_{n+1}-t_0]^{\alpha+1}}{\alpha+1}$$
$$-\frac{9h[t_{n+1}-t_{n+1}]^{\alpha+2}-[t_{n+1}-t_0]^{\alpha+2}}{\alpha+2}+\frac{[t_{n+1}-t_{n+1}]^{\alpha+3}-[t_{n+1}-t_0]^{\alpha+3}}{\alpha+3}]$$

$$=\frac{F_n}{6h^3}[\frac{-24h^3[-(n+1)^{\alpha}h^{\alpha}]}{\alpha}+\frac{26h^2[-(n+1)^{\alpha+1}h^{\alpha+1}]}{\alpha+1}-\frac{9h[-(n+1)^{\alpha+2}h^{\alpha+2}]}{\alpha+2}$$
$$+\frac{[-(n+1)^{\alpha+3}h^{\alpha+3}]}{\alpha+3}]$$

$$=h^{\alpha}[\frac{4(n+1)^{\alpha}}{\alpha}-\frac{13(n+1)^{\alpha+1}}{3(\alpha+1)}+\frac{3(n+1)^{\alpha+2}}{2(\alpha+2)}-\frac{(n+1)^{\alpha+3}}{6(\alpha+3)}]F_n$$

Similarly, for I_2, I_3 and I_4 we will have:

$$I_2=h^{\alpha}\left[\frac{-6(n+1)^{\alpha}}{\alpha}+\frac{19(n+1)^{\alpha+1}}{2(\alpha+1)}-\frac{4(n+1)^{\alpha+2}}{(\alpha+2)}+\frac{(n+1)^{\alpha+3}}{2(\alpha+3)}\right]F_{n-1}.$$

$$I_3=h^{\alpha}\left[\frac{4(n+1)^{\alpha}}{\alpha}-\frac{7(n+1)^{\alpha+1}}{(\alpha+1)}+\frac{7(n+1)^{\alpha+2}}{2(\alpha+2)}-\frac{(n+1)^{\alpha+3}}{2(\alpha+3)}\right]F_{n-2}.$$

$$I_4=h^{\alpha}\left[\frac{-(n+1)^{\alpha}}{\alpha}+\frac{11(n+1)^{\alpha+1}}{6(\alpha+1)}-\frac{(n+1)^{\alpha+2}}{(\alpha+2)}+\frac{(n+1)^{\alpha+3}}{6(\alpha+3)}\right]F_{n-3}.$$

Now, putting the values of I_1, I_2, I_3 and I_4 in equation (11) we get:

$$h^{\alpha}[(\frac{4(n+1)^{\alpha}}{\alpha}-\frac{13(n+1)^{\alpha+1}}{3(\alpha+1)}+\frac{3(n+1)^{\alpha+2}}{2(\alpha+2)}-\frac{(n+1)^{\alpha+3}}{6(\alpha+3)})F_n$$
$$+(\frac{-6(n+1)^{\alpha}}{\alpha}+\frac{19(n+1)^{(\alpha+1)}}{2(\alpha+1)}-\frac{4(n+1)^{\alpha+2}}{(\alpha+2)}+\frac{(n+1)^{\alpha+3}}{2(\alpha+3)})F_{n-1}$$
$$+(\frac{4(n+1)^{\alpha}}{\alpha}-\frac{7(n+1)^{(\alpha+1)}}{(\alpha+1)}+\frac{7(n+1)^{\alpha+2}}{2(\alpha+2)}-\frac{(n+1)^{\alpha+3}}{2(\alpha+3)})F_{n-2}$$
$$+(\frac{-(n+1)^{\alpha}}{\alpha}+\frac{11(n+1)^{(\alpha+1)}}{6(\alpha+1)}-\frac{(n+1)^{\alpha+2}}{(\alpha+2)}+\frac{(n+1)^{\alpha+3}}{6(\alpha+3)})F_{n-3}]. \quad \ldots(12)$$

The second integral in equation (10) is evaluated as:

$$\int_0^{t_n}(t_n-\lambda)^{\alpha-1}f\big(\lambda,y(\lambda)\big)d\lambda$$

$$=\sum_{p=0}^{n-1}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}\Big[\frac{(t-t_{n-1})(t-t_{n-2})(t-t_{n-3})}{(t_n-t_{n-1})(t_n-t_{n-2})(t_n-t_{n-3})}F_n+\frac{(t-t_n)(t-t_{n-2})(t-t_{n-3})}{(t_{n-1}-t_n)(t_{n-1}-t_{n-2})(t_{n-1}-t_{n-3})}F_{n-1}+\frac{(t-t_n)(t-t_{n-1})(t-t_{n-3})}{(t_{n-2}-t_n)(t_{n-2}-t_{n-1})(t_{n-2}-t_{n-3})}F_{n-2}+\frac{(t-t_n)(t-t_{n-1})(t-t_{n-2})}{(t_{n-3}-t_n)(t_{n-3}-t_{n-1})(t_{n-3}-t_{n-2})}F_{n-3}\Big]dt.$$

$$=\sum_{p=0}^{n-1}\Big[\frac{F_n}{(t_n-t_{n-1})(t_n-t_{n-2})(t_n-t_{n-3})}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_{n-1})(t-t_{n-2})(t-t_{n-3})dt$$
$$+\frac{F_{n-1}}{(t_{n-1}-t_n)(t_{n-1}-t_{n-2})(t_{n-1}-t_{n-3})}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_n)(t-t_{n-2})(t-t_{n-3})dt$$
$$+\frac{F_{n-2}}{(t_{n-2}-t_n)(t_{n-2}-t_{n-1})(t_{n-2}-t_{n-3})}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_n)(t-t_{n-1})(t-t_{n-3})dt$$
$$+\frac{F_{n-3}}{(t_{n-3}-t_n)(t_{n-3}-t_{n-1})(t_{n-3}-t_{n-2})}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_n)(t-t_{n-1})(t-t_{n-2})dt\Big].$$

$$=\sum_{p=0}^{n-1}\Big[\frac{F_n}{6h^3}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_{n-1})(t-t_{n-2})(t-t_{n-3})dt$$
$$-\frac{F_{n-1}}{2h^3}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_n)(t-t_{n-2})(t-t_{n-3})dt$$
$$+\frac{F_{n-2}}{2h^3}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_n)(t-t_{n-1})(t-t_{n-3})dt$$
$$-\frac{F_{n-3}}{6h^3}\int_{t_p}^{t_{p+1}}(t_n-t)^{\alpha-1}(t-t_n)(t-t_{n-1})(t-t_{n-2})dt\Big].$$

$$=\sum_{p=0}^{n-1}\Big\{\frac{F_n}{6h^3}\left[\int_{t_n-t_p}^{t_n-t_{p+1}}y^{\alpha-1}(t_n-y-t_{n-1})(t_n-y-t_{n-2})(t_n-y-t_{n-3})(-dy)\right]$$
$$-\frac{F_{n-1}}{2h^3}\left[\int_{t_n-t_p}^{t_n-t_{p+1}}y^{\alpha-1}(t_n-y-t_n)(t_n-y-t_{n-2})(t_n-y-t_{n-3})(-dy)\right]$$
$$+\frac{F_{n-2}}{2h^3}\left[\int_{t_n-t_p}^{t_n-t_{p+1}}y^{\alpha-1}(t_n-y-t_n)(t_n-y-t_{n-1})(t_n-y-t_{n-3})(-dy)\right]$$
$$-\frac{F_{n-3}}{6h^3}\Big[\int_{t_n-t_p}^{t_n-t_{p+1}}y^{\alpha-1}(t_n-y-t_n)(t_n-y-t_{n-1})(t_n-y-t_{n-2})(-dy)\Big]\Big\}. \qquad \text{... (13)}$$

Let,

$$I'_1=\sum_{p=0}^{n-1}\Big\{\frac{F_n}{6h^3}\Big[\int_{t_n-t_p}^{t_n-t_{p+1}}y^{\alpha-1}(t_n-y-t_{n-1})(t_n-y-t_{n-2})(t_n-y-t_{n-3})(-dy)\Big]\Big\}.$$

$$I'_2 = \sum_{p=0}^{n-1}\{-\frac{F_{n-1}}{2h^3}[\int_{t_n-t_p}^{t_n-t_{p+1}} y^{\alpha-1}(t_n-y-t_n)(t_n-y-t_{n-2})(t_n-y-t_{n-3})(-dy)]\}.$$

$$I'_3 = \sum_{p=0}^{n-1}\{\frac{F_{n-2}}{2h^3}[\int_{t_n-t_p}^{t_n-t_{p+1}} y^{\alpha-1}(t_n-y-t_n)(t_n-y-t_{n-1})(t_n-y-t_{n-3})(-dy)]\}.$$

$$I'_4 = \sum_{p=0}^{n-1}\{-\frac{F_{n-3}}{6h^3}[\int_{t_n-t_p}^{t_n-t_{p+1}} y^{\alpha-1}(t_n-y-t_n)(t_n-y-t_{n-1})(t_n-y-t_{n-2})(-dy)]\}.$$

The computation for I'_1 is given as:

$$= \sum_{p=0}^{n-1}\{-\frac{F_n}{6h^3}[\int_{t_n-t_p}^{t_n-t_{p+1}} y^{\alpha-1}(h-y)(2h-y)(3h-y)dy]\}.$$

$$= \sum_{p=0}^{n-1}\{-\frac{F_n}{6h^3}[\int_{t_n-t_p}^{t_n-t_{p+1}} y^{\alpha-1}(6h^3-11h^2y+6hy^2-y^3)dy]\}.$$

$$= \sum_{p=0}^{n-1}\{-\frac{F_n}{6h^3}\left[\int_{t_n-t_p}^{t_n-t_{p+1}}(6h^3y^{\alpha-1}-11h^2y^{\alpha}+6hy^{\alpha+1}-y^{\alpha+2})\,dy\right].$$

$$= \sum_{p=0}^{n-1}\{-\frac{F_n}{6h^3}[\frac{6h^3y^{\alpha}}{\alpha}-\frac{11h^2y^{\alpha+1}}{\alpha+1}+\frac{6hy^{\alpha+2}}{\alpha+2}-\frac{y^{\alpha+3}}{\alpha+3}]|_{t_n-t_p}^{t_n-t_{p+1}}\}.$$

$$= \sum_{p=0}^{n-1}\{-\frac{F_n}{6h^3}[\frac{6h^3[t_n-t_{p+1}]^{\alpha}-[t_n-t_p]^{\alpha}}{\alpha}-\frac{11h^2[t_n-t_{p+1}]^{\alpha+1}-[t_n-t_p]^{\alpha+1}}{\alpha+1}$$
$$+\frac{6h[t_n-t_{p+1}]^{\alpha+2}-[t_{n+1}-t_p]^{\alpha+2}}{\alpha+2}-\frac{[t_n-t_{p+1}]^{\alpha+3}-[t_n-t_p]^{\alpha+3}}{\alpha+3}]\}.$$

$$= -[\frac{F_n}{6h^3}\frac{6h^3[t_n-t_n]^{\alpha}-[t_n-t_0]^{\alpha}}{\alpha}-\frac{11h^2[t_n-t_n]^{\alpha+1}-[t_n-t_0]^{\alpha+1}}{\alpha+1}$$
$$+\frac{6h[t_n-t_n]^{\alpha+2}-[t_n-t_0]^{\alpha+2}}{\alpha+2}-\frac{[t_n-t_n]^{\alpha+3}-[t_n-t_0]^{\alpha+3}}{\alpha+3}].$$

$$= -\frac{F_n}{6h^3}[\frac{6h^3[-(n)^{\alpha}h^{\alpha}]}{\alpha}-\frac{11h^2[-(n)^{\alpha+1}h^{\alpha+1}]}{(\alpha+1)}+\frac{6h[-(n)^{\alpha+2}h^{\alpha+2}]}{\alpha+2}$$
$$-\frac{[-(n)^{\alpha+3}h^{\alpha+3}]}{\alpha+3}].$$

$$= h^{\alpha}\left[\frac{(n)^{\alpha}}{\alpha}-\frac{11(n)^{\alpha+1}}{6(\alpha+1)}+\frac{(n)^{\alpha+2}}{\alpha+2}-\frac{(n)^{\alpha+3}}{6(\alpha+3)}\right]F_n.$$

Similarly, for I'_2, I'_3 and I'_4 we will have:

$$I'_2 = h^{\alpha}\left[\frac{3(n)^{\alpha+1}}{(\alpha+1)}-\frac{5(n)^{\alpha+2}}{2(\alpha+2)}+\frac{(n)^{\alpha+3}}{2(\alpha+3)}\right]F_{n-1}.$$

$$I'_3 = h^{\alpha}\left[-\frac{3(n)^{\alpha+1}}{2(\alpha+1)}+\frac{2(n)^{\alpha+2}}{(\alpha+2)}-\frac{(n)^{\alpha+3}}{2(\alpha+3)}\right]F_{n-2}.$$

$$I'_4 = h^{\alpha}\left[\frac{(n)^{\alpha+1}}{3(\alpha+1)} - \frac{(n)^{\alpha+2}}{2(\alpha+2)} + \frac{(n)^{\alpha+3}}{6(\alpha+3)}\right]F_{n-3}.$$

Now, putting the values of I'_1, I_2, I'_3 and I'_4 in (13) we get:

$$\begin{aligned} &h^{\alpha}[(\frac{(n)^{\alpha}}{\alpha} - \frac{11(n)^{\alpha+1}}{6(\alpha+1)} + \frac{(n)^{\alpha+2}}{\alpha+2} - \frac{(n)^{\alpha+3}}{6(\alpha+3)})F_n \\ &+ (\frac{3(n)^{\alpha+1}}{(\alpha+1)} - \frac{5(n)^{\alpha+2}}{2(\alpha+2)} + \frac{(n)^{\alpha+3}}{2(\alpha+3)})F_{n-1} \\ &- (\frac{3(n)^{\alpha+1}}{2(\alpha+1)} - \frac{2(n)^{\alpha+2}}{(\alpha+2)} + \frac{(n)^{\alpha+3}}{2(\alpha+3)})F_{n-2} \\ &+ (\frac{(n)^{\alpha+1}}{3(\alpha+1)} - \frac{(n)^{\alpha+2}}{2(\alpha+2)} + \frac{(n)^{\alpha+3}}{6(\alpha+3)})F_{n-3}]. \qquad \dots (14) \end{aligned}$$

Substituting equation (12) and equation (14) in equation (10) gives the following fractional four step explicit Adams Bashforth formula:

$$\begin{aligned} y(t_{n+1}) = y(t_n) + \frac{h^{\alpha}}{\Gamma(\alpha)}\{[&\frac{4(n+1)^{\alpha} - n^{\alpha}}{\alpha} + \frac{11(n)^{\alpha+1} - 26(n+1)^{\alpha+1}}{6(\alpha+1)} + \frac{3(n+1)^{\alpha+2} - 2(n)^{\alpha+2}}{2(\alpha+2)} \\ &+ \frac{(n)^{\alpha+3} - (n+1)^{\alpha+3}}{6(\alpha+3)}]F_n + \\ &[\frac{-6(n+1)^{\alpha}}{\alpha} + \frac{19(n+1)^{\alpha+1} - 6(n)^{\alpha+1}}{2(\alpha+1)} + \frac{5(n)^{\alpha+2} - 8(n+1)^{\alpha+2}}{2(\alpha+2)} \\ &+ \frac{(n+1)^{\alpha+3} - (n)^{\alpha+3}}{2(\alpha+3)}]F_{n-1} + \\ &[\frac{4(n+1)^{\alpha}}{\alpha} + \frac{3(n)^{\alpha+1} - 14(n+1)^{\alpha+1}}{2(\alpha+1)} + \frac{7(n+1)^{\alpha+2} - 4(n)^{\alpha+2}}{2(\alpha+2)} \\ &+ \frac{(n)^{\alpha+3} - (n+1)^{\alpha+3}}{2(\alpha+3)}]F_{n-2} + \\ &[\frac{-(n+1)^{\alpha}}{\alpha} + \frac{11(n+1)^{\alpha+1} - 2(n)^{\alpha+1}}{6(\alpha+1)} + \frac{(n)^{\alpha+2} - 2(n+1)^{\alpha+2}}{2(\alpha+2)} \\ &+ \frac{(n+1)^{\alpha+3} - (n)^{\alpha+3}}{6(\alpha+3)}]F_{n-3}\}. \end{aligned}$$

Remark: For α =1 we recover the classical 4 step Adams Bashforth explicit Numerical scheme.

4. Applications

Example 1:

: Consider the linear fractional differential equation:

$$D_t^{\alpha}u(t) = -u(t) + t^{3+\alpha} + \frac{\Gamma[4+\alpha]}{6}t^3, \quad 0 < \alpha < 1,$$

Along with exact solution of this equation given by:

$$u(t) = t^{3+\alpha}.$$

If we apply the derived FFSEABM by taking $0 < t \leq 1$, step size $N = 100$ and initial condition $u(0) = 0$, we can obtain numerical solution for the FDE. For the first 15 terms exact solutions, approximate solutions and absolute errors are given in the following table.

Table 1: For Example 1, Approximated Solutions and Absolute Errors at each point t when α = 0.5 and N=100

i	t	y_exact	y_approx	Absolute Error
1	0.00	0.0000000e+00		
2	0.01	1.0000000e-07		
3	0.02	1.1313708e-06		
4	0.03	4.6765372e-06		
5	0.04	1.2800000e-05	1.2801551e-05	1.5509268e-09
6	0.05	2.7950850e-05	2.7953093e-05	2.2428311e-09
7	0.06	5.2908978e-05	5.2912221e-05	3.2420765e-09
8	0.07	9.0749270e-05	9.0752798e-05	3.5280758e-09
9	0.08	1.4481547e-04	1.4481963e-04	4.1619561e-09
10	0.09	2.1870000e-04	2.1870418e-04	4.1786874e-09
11	0.10	3.1622777e-04	3.1623257e-04	4.8033617e-09
12	0.11	4.4144276e-04	4.4144722e-04	4.4625563e-09
13	0.12	5.9859676e-04	5.9860231e-04	5.5511570e-09
14	0.13	7.9213962e-04	7.9214365e-04	4.0298954e-09
15	0.14	1.0267108e-03	1.0267184e-03	7.6000785e-09

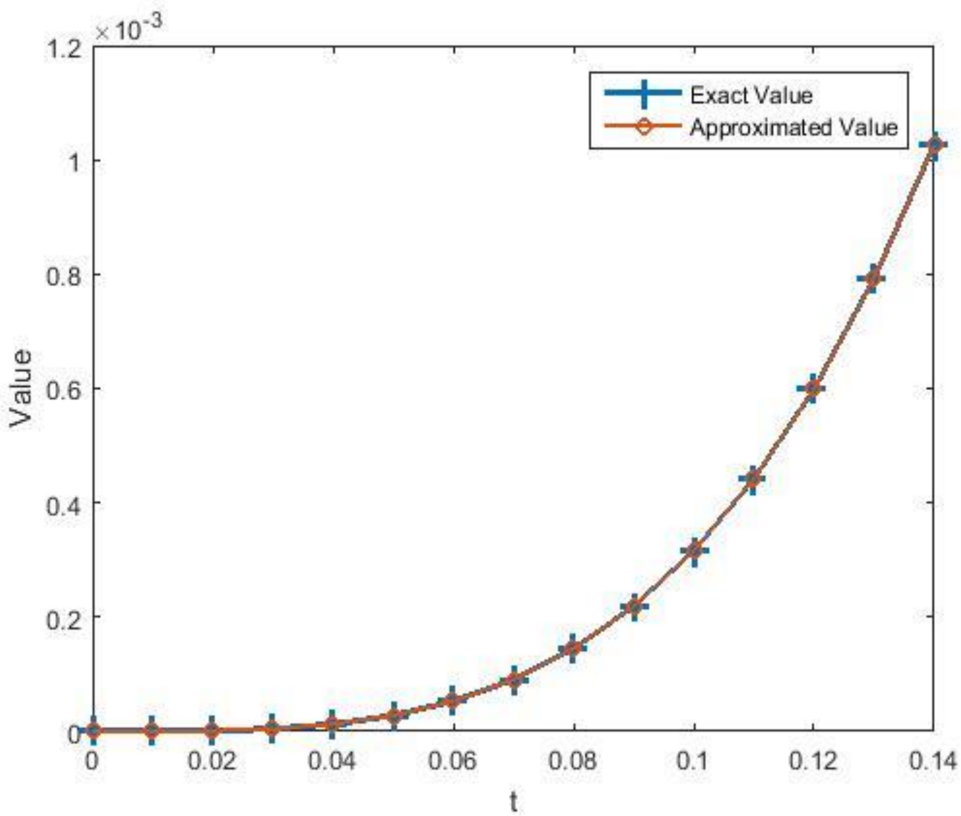

Figure 1: For Example 1, Plot between approximated and exact value for α = 0.5

For α = 0.7 we can obtain the following numerical solutions at some grid points.

Table 2: For example 1, Approximated Solutions and Absolute Errors at each point t when α = 0.7 and N=100

i	t	y_exact	y_approx	Absolute Error
1	0.00	0.0000000e+00		
2	0.01	3.9810717e-08		
3	0.02	5.1738161e-07		
4	0.03	2.3192558e-06		
5	0.04	6.7239112e-06	6.7248075e-06	8.9632883e-10
6	0.05	1.5352850e-05	1.5354333e-05	1.4822399e-09
7	0.06	3.0141137e-05	3.0143155e-05	2.0181520e-09
8	0.07	5.3316637e-05	5.3319018e-05	2.3805804e-09
9	0.08	8.7384207e-05	8.7386909e-05	2.7016966e-09
10	0.09	1.3511305e-04	1.3511599e-04	2.9436762e-09
11	0.10	1.9952623e-04	1.9952940e-04	3.1674603e-09
12	0.11	2.8389186e-04	2.8389520e-04	3.3401728e-09
13	0.12	3.9171537e-04	3.9171888e-04	3.5085216e-09
14	0.13	5.2673285e-04	5.2673648e-04	3.6305583e-09
15	0.14	6.9290506e-04	6.9290883e-04	3.7684438e-09

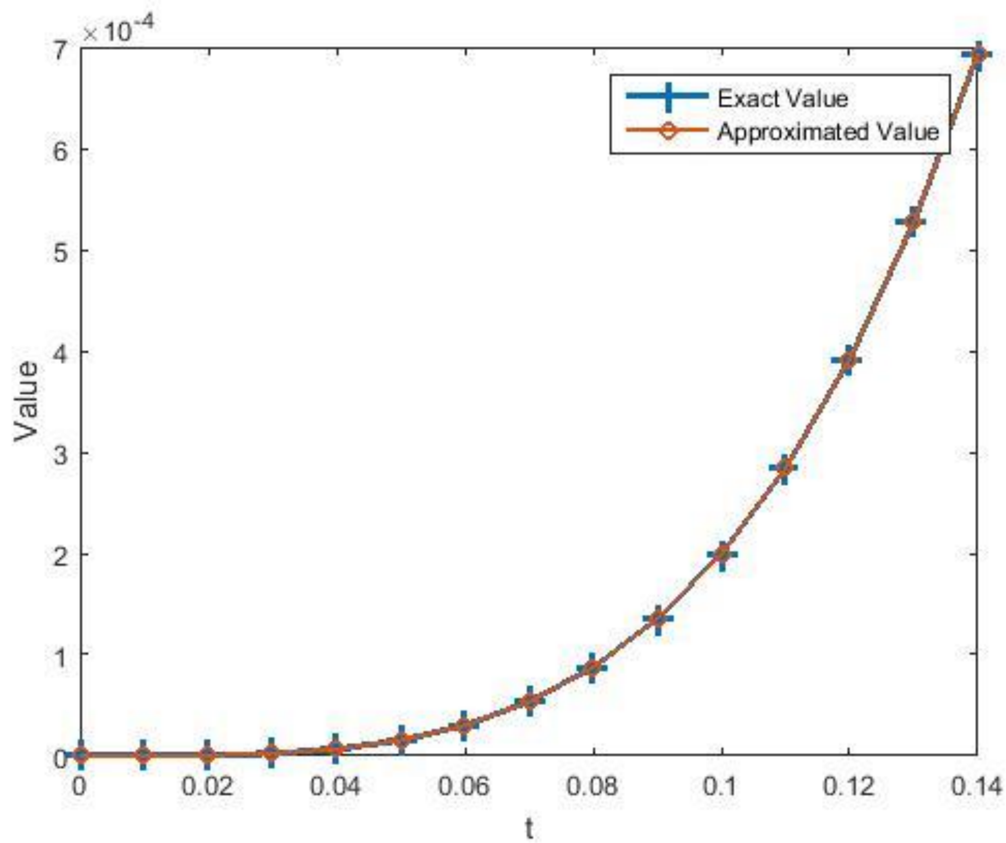

Figure 2: For example, 1, Plot between approximated and exact value for α = 0.7

Example 2: Consider the non-linear fractional differential equation:

$$D_t^{\alpha} u(t) = -u^2(t) + \frac{\Gamma[5+\alpha]}{24} t^4 + t^{8+2\alpha} \qquad 0 < \alpha \le 1.$$

Exact solution of this equation is given by: $u(t) = t^{4+\alpha}$.

If we apply the derived FFSEABM by taking $0 < t \le 1$, step size $N = 100$ and initial condition $u(0) = 0$, we obtain numerical solution for the FIVP.

Table 3: For example 2, Approximated Solutions and Absolute Errors at each point t when α = 0.5 and N=100

i	t	y_exact	y_approx	Absolute Error
1	0.00	0.0000000e+00		
2	0.01	1.0000000e-09		
3	0.02	2.2627417e-08		
4	0.03	1.4029612e-07		
5	0.04	5.1200000e-07	4.7987450e-07	3.2125504e-08
6	0.05	1.3975425e-06	1.3366944e-06	6.0848107e-08
7	0.06	3.1745387e-06	3.0942352e-06	8.0303522e-08
8	0.07	6.3524489e-06	6.2716423e-06	8.0806636e-08
9	0.08	1.1585238e-05	1.1537025e-05	4.8212655e-08
10	0.09	1.9683000e-05	1.9719645e-05	3.6644642e-08
11	0.10	3.1622777e-05	3.1821235e-05	1.9845882e-07
12	0.11	4.8558704e-05	4.9026618e-05	4.6791403e-07
13	0.12	7.1831611e-05	7.2713725e-05	8.8211423e-07
14	0.13	1.0297815e-04	1.0446313e-04	1.4849842e-06
15	0.14	1.4373951e-04	1.4606716e-04	2.3276465e-06

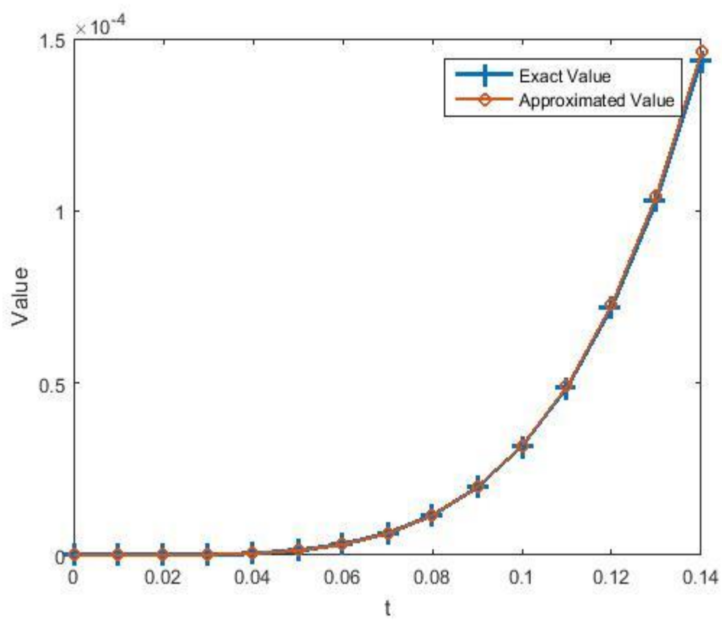

Figure 3: For example, 2, Plot between approximated and exact value for α = 0.5

For $\alpha = 0.75$ we obtain the solution as follow:

Table 4: For example 2, Approximated Solutions and Absolute Errors at each point t when α = 0.7 and N= 100

i	t	y_exact	y_approx	Absolute Error
1	0.00	0.0000000e+00		
2	0.01	3.9810717e-10		
3	0.02	1.0347632e-08		
4	0.03	6.9577673e-08		
5	0.04	2.6895645e-07	2.5481300e-07	1.4143449e-08
6	0.05	7.6764252e-07	7.4148267e-07	2.6159846e-08
7	0.06	1.8084682e-06	1.7762923e-06	3.2175857e-08
8	0.07	3.7321646e-06	3.7066219e-06	2.5542708e-08
9	0.08	6.9907366e-06	6.9945089e-06	3.7723029e-09
10	0.09	1.2160174e-05	1.2229927e-05	6.9752710e-08
11	0.10	1.9952623e-05	2.0143491e-05	1.9086767e-07
12	0.11	3.1228104e-05	3.1618685e-05	3.9058095e-07
13	0.12	4.7005844e-05	4.7703683e-05	6.9783863e-07
14	0.13	6.8475270e-05	6.9622809e-05	1.1475385e-06
15	0.14	9.7006708e-05	9.8787692e-05	1.7809837e-06

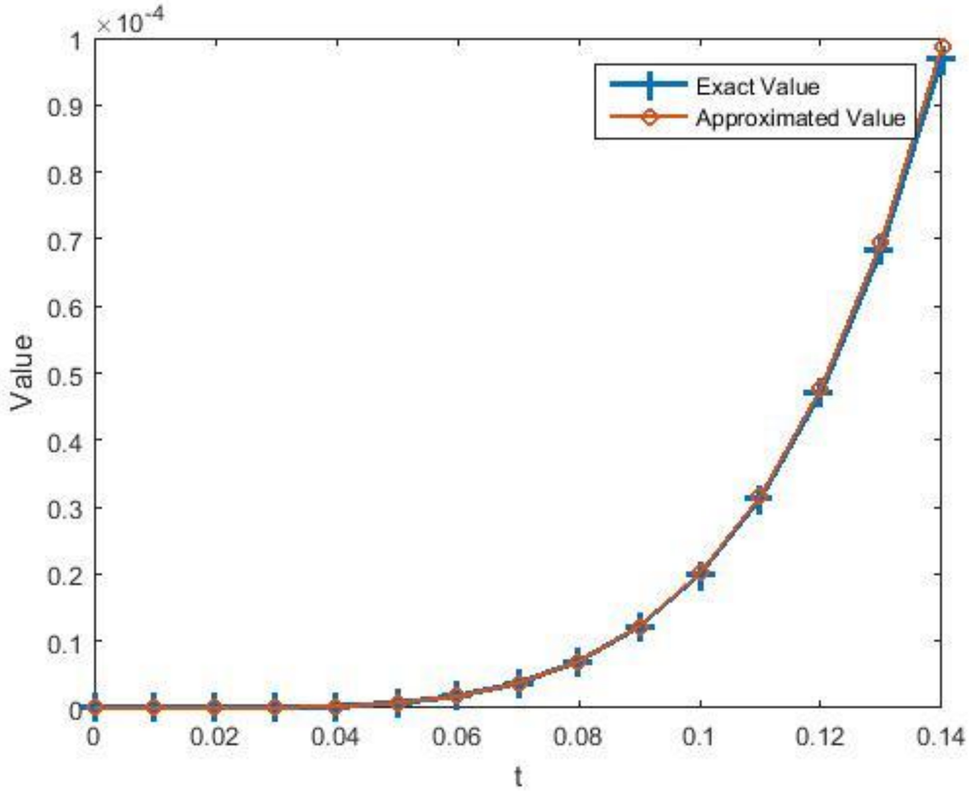

Figure 4: For example, 2, Plot between approximated and exact value for α = 0.7

In each example we are using the exact values given by the exact solution as starting values. The tables show the exact solutions of the given Fractional differential equations at different values of t along with their approximated answers using the proposed algorithm. The small absolute error shows that the proposed method is convergent. Moreover, the graphs also demonstrate that the exact and approximated answers are very close.

5. Discussion

The aim of the research was to provide an accurate and efficient numerical method for solving FDEs which are important in various areas of science and engineering. The fractional four step explicit Adams Bashforth method and its implementation for solving FDEs were then presented. The results showed that the method has high accuracy and computational efficiency, making it a promising numerical method for solving these types of equations. This research provided a valuable contribution to the field of numerical methods for fractional differential equations, demonstrating the effectiveness of the fractional explicit Adams method.

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