

ESTIMATING THE ERROR TERM IN THE TRAPEZIUM RULE USING A RUNGE-KUTTA METHOD

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Abstract

We show how the error term for the Trapezium Rule can be estimated, by solving an initial value problem using a Runge-Kutta method. The error term can then be added to the Trapezium approximation, yielding a much more accurate result. We also show how the risk of singularities in the relevant initial value problem can be mitigated.



1. Introduction

Scholars of applied mathematics, and numerical analysis in particular, have no doubt encountered the Trapezium Rule (Burden & Faires, 2011; Kincaid & Cheney, 2002) and Runge-Kutta (RK) methods (Butcher, 2003). They will very likely have met the error term for the Trapezium Rule, and they may even know how to estimate an integral using an RK method. However, they may not know that the

$$\int_a^x f(x) dx = \frac{(x-a)}{2} (f(a) + f(x)) - \frac{(x-a)^3}{12} f''(\xi(x))$$

$$\xi \in (a, x).$$
(1)

Here, $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be Riemann integrable, the upper limit in the integral is variable, the first term on the RHS is the Trapezium Rule written in terms of x ; and the

Trapezium Rule's error term can be estimated by solving a suitable initial value problem using, of course, an RK method. Once the error term is known, the Trapezium approximation can be considerably improved. This interesting and potentially useful device is the topic of this paper.

2. Theory

We have the equation

second term on the RHS is the error in the Trapezium Rule, wherein we assume that is a function of x . Differentiating wrt x gives

$$f(x) = \frac{1}{2} (f(a) + f(x)) + \frac{(x-a)f'(x)}{2} - \frac{3(x-a)^2}{12} f''(\xi) - \frac{(x-a)^3}{12} \frac{df''(\xi)}{d\xi} \frac{d\xi}{dx}$$

which yields the differential equation (DE)

$$\frac{d\xi}{dx} = \frac{-18f(x) + 6f(a) + 6(x-a)f'(x) - 3(x-a)^2 f''(\xi)}{(x-a)^3 f'''(\xi)} : \quad (2)$$

If an initial value (x_0) is provided, this DE can be solved using a Runge-Kutta method to yield (x): This allows the error term to be computed, which can then be added to the Trapezium term to obtain a more accurate approximation for the integral. Note that we must necessarily assume

$$\int_1^x \sin x dx.$$

This gives

$$\frac{d\xi}{dx} = \frac{-18 \sin x + 6 \sin(1) + 6(x-1) \cos x + 3(x-1)^2 \sin \xi}{-(x-1)^3 \cos \xi} : \quad (3)$$

We will consider an upper limit of $x = 10$ here. To find an initial value, we choose $x_0 = 2$ (1;10); such that x_0 is not too close to 1 (to avoid

that f and its first three derivatives all exist on $[a;x]$:

3. Example

To illustrate this method, we consider the simple example

potential problems in the denominator in (3)).

We substitute x_0 into (1) to get

$$\frac{(x_0-1)}{2} (\sin(1) + \sin x_0) + \frac{(x_0-1)^3}{12} \sin(\xi(x_0)) - \int_1^{x_0} \sin x dx = 0 \quad (4)$$

Which we solve for (x_0) using a numerical method such as Newton's Method or the Bisection Method. Of course, we will have to invest some effort in obtaining an accurate value for $\int_1^{x_0} \sin x dx$, which can be achieved using the *composite* Trapezium Rule, for example. This effort will be well rewarded. Once $x_0 = (x_0)$ has been determined, we solve (3) subject to the initial value ($x_0; 0$); from x_0 up to

10; and then from x_0 down to 1; using a suitable RK method. We use a seventh-order method (RK7) (Butcher, 2003) in this work. We provide an insight into the reverse RK process from x_0 down to 1 in the Appendix. We choose $x_0 = 5$: We find, using composite Trapezium quadrature (Kincaid & Cheney, 2002), $\int_1^5 \sin x dx = 0.256640120404911$; accurate to 10⁻¹⁵. Hence, we find $x_0 =$

3:049296665128674 using the Bisection Method (Burden & Faires, 2011), also to an

accuracy of 10^{-15} : Using RK7 to solve the DE gives the results shown in the figures.

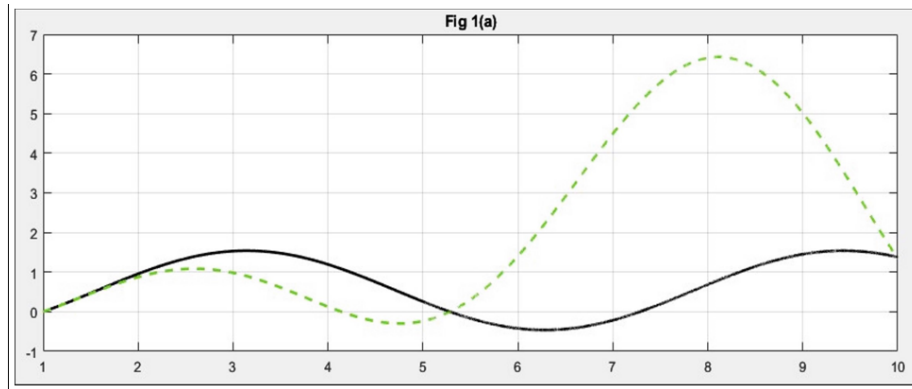


Figure 1(a): Value of the integral and the Trapezium approximation

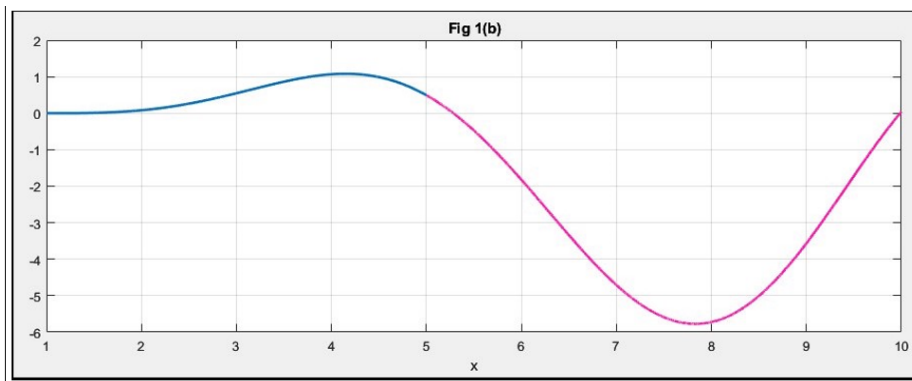


Figure 1(b): The error in the Trapezium Rule

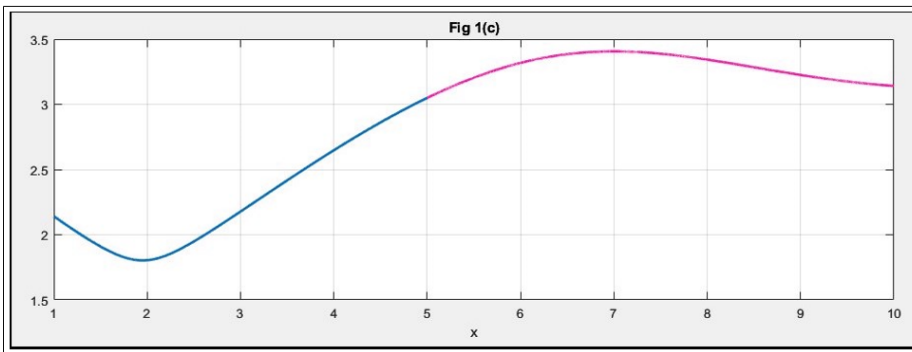


Figure 1(c): Riemann

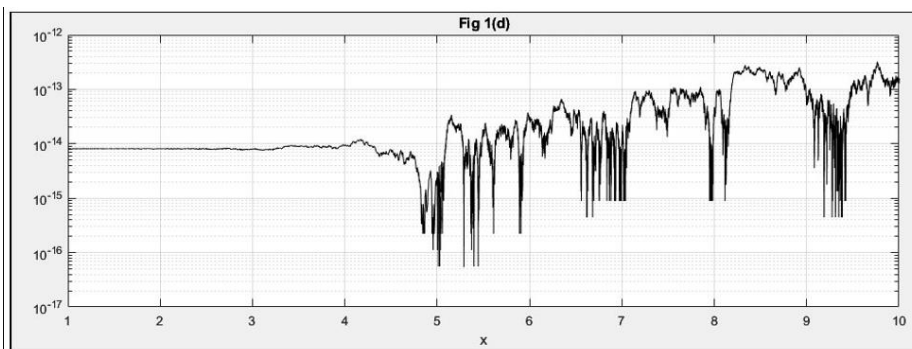


Figure 1(d): Quantity

In Figure 1(a), we show the true value of the integral and the Trapezium approximation (dotted line), vs x . Clearly, the Trapezium Rule is inaccurate. In Figure 1(b), we show the error

in the Trapezium Rule vs x ; where we see that the error is, in places, larger than the integral itself. In Figure 1(c), we show (x) : In Figure 1(d), we show the quantity,

$$\left| \int_1^x \sin x dx - \left(\underbrace{\frac{(x-1)}{2} (\sin(1) + \sin x)}_{\text{Trapezium Rule}} + \underbrace{\frac{(x-1)^3}{12} \sin(\xi(x))}_{\text{Error term}} \right) \right|. \quad (5)$$

It is clear that adding the error term to the Trapezium approximation yields values that are considerably more accurate. In a sense, we have used the error term as a correction term. Considering the more exotic integrand $f(x) = x^2(\sin x \ln(2+x) - 100x)$; which is not analytically integrable, and using the same limits and the same x_0 as above, we find the error in the Trapezium Rule is as high as 2×10^5 : However, when we add the correction term the error is reduced to 10^{-10} ; a remarkable improvement indeed.

4. Conclusion

We have shown how the error term for the Trapezium Rule can be estimated, by solving an appropriate initial value problem using a Runge-Kutta method. Adding the error term to the Trapezium approximation yields significantly more accurate results. Naturally, the accuracy that is attained is dependent on the accuracy of the Runge-Kutta solution. In our example, we used a small enough stepsize so that we achieved a high degree of accuracy. Ordinarily, we would not know the required stepsize beforehand, and we would implement the Runge-Kutta method with some form of error control- although not doing that here does not detract from our central result.

References

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